NATIONAL MATHEMATICS MAGAZINE

(Formerly Mathematics News Letter)

Vol. XII BATON ROUGE, LA., OCTOBER, 1937 No. 1

PUBLISHED BY LOUISIANA STATE UNIVERSITY

Every paper on technical mathematics offered for publication should be submitted (with enough enclosed postage to cover two two-way transmissions) to the Chairman of the appropriate Committee, or to a Committee member whom the Chairman may designate to examine it, after being requested to do so by the writer. If approved for publication, the Committee will forward it to the Editor and Manager at Baton Rouge, who will notify the writer of its acceptance for publication. If the paper is not approved the Committee will so notify the Editor and Manager, who will inform the writer accordingly.

1. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

2. The name of the Chairman of each committee is the first in the list of the committee.

3. All manuscripts should be worded exactly as the author wishes them to appear in the Magazine.

Papers intended for the Teacher's Department, Department of History of Mathematics, Reviews and Abstracts or Problem Department should be sent to the respective Chairmen.

Published by the Louisiana State University Press

Club Rates
May Be Had
on Application

Subscription, $1.50
Per Year in Advance
Single Copies, 20c.

Vol. XII BATON ROUGE, LA., OCTOBER, 1937 No. 1

Published 8 Times Each Year by Louisiana State University. Vols. 1-8 Published as Mathematics News Letter.

All Business Communications should be addressed to the Editor and Manager P. O. Box 1322, Baton Rouge, La.

This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Taking Issue

To the Editor of the National Mathematics Magazine

Sir:

The following statement in your article, "*Quality and Mathematics*", in your May issue, is difficult to reconcile with the state of science, philosophy, and the arts today:

> To assert that the *quality* of a thing is not amenable to mathematical process is to assert the existence of domains of reality of whose nature and properties a knowledge may be acquired *without* the aids of precise definition logical deduction, or clarified and classified concepts. All of these are the very inner essence of mathematics, properly conceived. Knowledge without definiteness is no knowledge at all.

There are non-mathematical fields where precise definition, logical deduction, and clarified and classified concepts are indispensable, such as logic, the law, biological science, morals, public affairs. Are these not "properly conceived"? Supposing they are not, but hypothetically might be, would they then be wholly absorbed by mathematics? The stake which mathematics can claim in these fields must evidently remain slight. The statement that knowledge without definiteness is no knowledge at all calls to mind, on the contrary, the sense of uncertainty which at present characterizes large sections of science and some of mathematics. A recent article by a member of the Viennese Circle represents the whole body of science as a ship on the open sea. (Otto Naurath: "*Philosophy of Science*", Vol. 4, p. 276.) This is the view of the Circle as a whole, and I may note that this group comprises not laymen but mathematicians, logicians, epistemologists, and scientific men.

Elsewhere in your article occurs the statement: "Quality is a definite answer to nothing." It may be objected that in some spheres a *definite* answer is impossible without distortion and is furthermore undesired; for example, poetry, the arts, the whole realm of axiological values. The fruitless results of scientific, including mathematical, experiment in aesthetics is illuminating here.

But the following from your last paragraph raises the most serious questions of all:

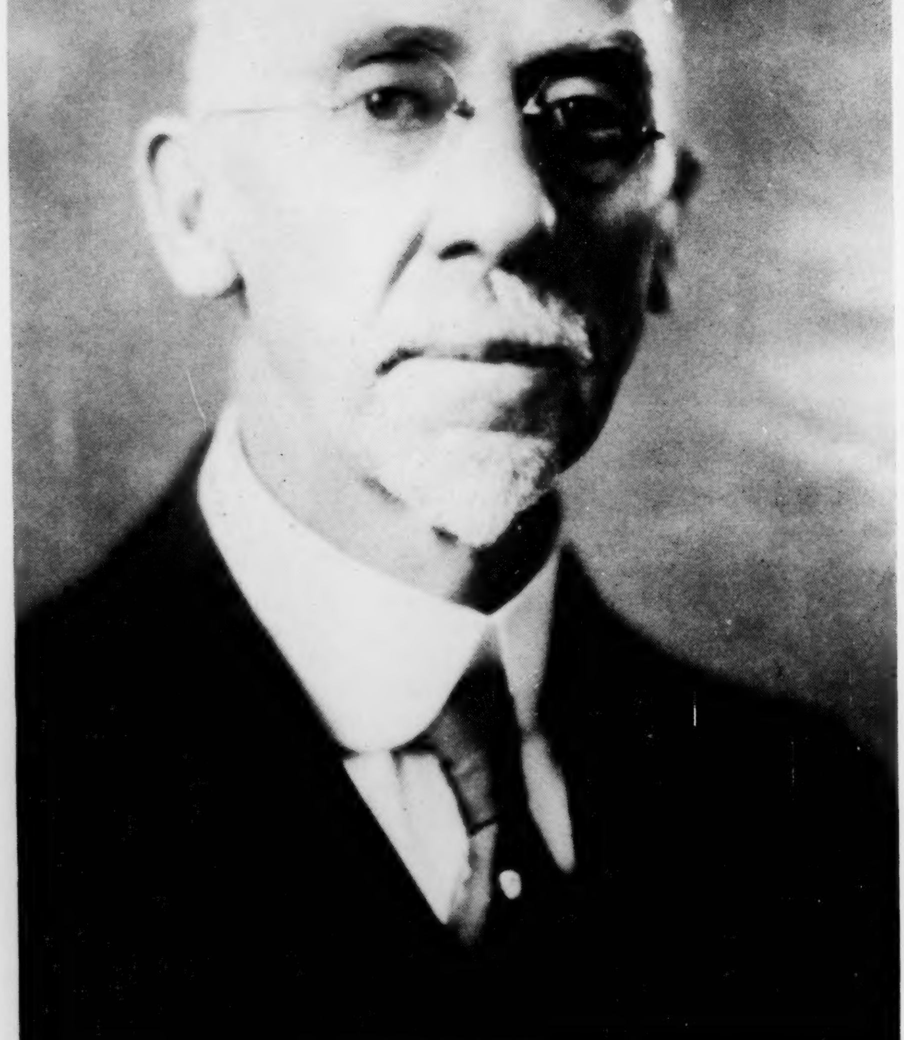
> Not until an entity, or a state of being of an entity, or a quality of an entity has been identified by *defining* characteristics, can it be made a subject of proper thought. Once it has been identified in such manner, mathematics, and mathematics alone, is the carrier ordained to bring the answers to questions without which *all notions* of quality must be vain and useless.

If by *proper* thought you mean *mathematical* thought, are you not arbitrarily excluding all other thought? If on the other hand this is not an arbitrary qualification, then the objection must be met which holds (with support from psychology and records of the creative experience) that a great amount, perhaps a great majority, of human thought is not in terms of entities "Identified by *defining* characteristics". Also, the proposition that mathematics is *ordained* in the manner stated raises doubts and suspicions. By what or by whom *ordained*?

I should think the views of the writer you were criticizing, which were that the realm of mathematics is essentially beyond concrete affairs, were sound both theoretically and historically. They make mathematics independent of practical pursuits and allow for things wholly impractical yet logically valid, such as non-Euclidean geometry. Is not mathematics more lofty in the realm of pure reason than it could possibly be in the world of mere practice?

PETER A. CARMICHAEL.

Louisiana State University, June 23, 1937.



HERBERT ELLSWORTH SLAUGHT

Herbert Ellsworth Slaught

On May 21, 1937, Herbert Ellsworth Slaught died at his home in Chicago, near the University of Chicago, the institution to which had been dedicated forty years of his effective life.

The writer's personal knowledge of him dates back to more than a quarter of a century ago. We first sat in his classes as a student in the spring of 1908. From the very beginning, we were impressed with the dynamic humanness of the man. But little study of his teaching was needed to show that his power as a teacher of mathematics came essentially from the indivisible union in his nature of two things: (a) a profound love of mathematics, (b) an equally profound interest in his fellows, especially young people. These two qualities could scarcely have been consistent with any programs he might originally have entertained for extensive mathematical research. Indeed, the cause of mathematics in America would probably have suffered had he chosen to confine his energies to research. By temperament, he must find his greatest happiness in the work of an evangel acclaiming before his fellows the power and the beauty of mathematics; its glory as the base of our civilization; its all-pervasiveness, as, like the sunlight, it penetrates every nook where dwells the possibility of applied human reason.

The same elements of his nature that made him an inspiring teacher of class-room mathematics contributed most consistently to make of him the greatest organizer of large-scale group activities in mathematics that America has yet produced. For a more detailed account of his organizing genius, one may read page 23 of an article by Professor G. A. Bliss, published in the July, 1937, issue of the University of Chicago Magazine.

Professor Slaught's manner was entirely unpretentious and free from the slightest affectation. Wholesome humor was one of his marked characteristics, and was reflected in his

laugh. That laugh, utterly free from sarcasm or rancor, was so hearty, so sincere, so warm with the spirit of good-fellowship that quite every one to whom he might talk, whether in casual conversation or sitting as a student of one of his classes, could not avoid thinking of him more as a good pal than as a professor of mathematics.

Mathematicians as a rule are poor correspondents. Here is not the place to say why. Throughout many years of the writer's correspondence with Professor Slaught--a correspondence maintained largely in connection with the activities of the Louisiana-Mississippi section of M. A. of A.--no letter, note, or question was ever addressed to him that failed to have answer. It is our feeling that similar testimony could be truthfully furnished by hundreds of others who have had occasion to write to him about matters involving some service that Professor Slaught might render.

Finally, we profoundly honor the memory of Herbert Ellsworth Slaught in that, (1) he was a MAN in every real sense before he was a MATHEMATICIAN, (2) he so loved his mathematics, that, loving his fellowman also, he could only give his life to preaching its gospel to all his fellows, so that they too might share its blessings.

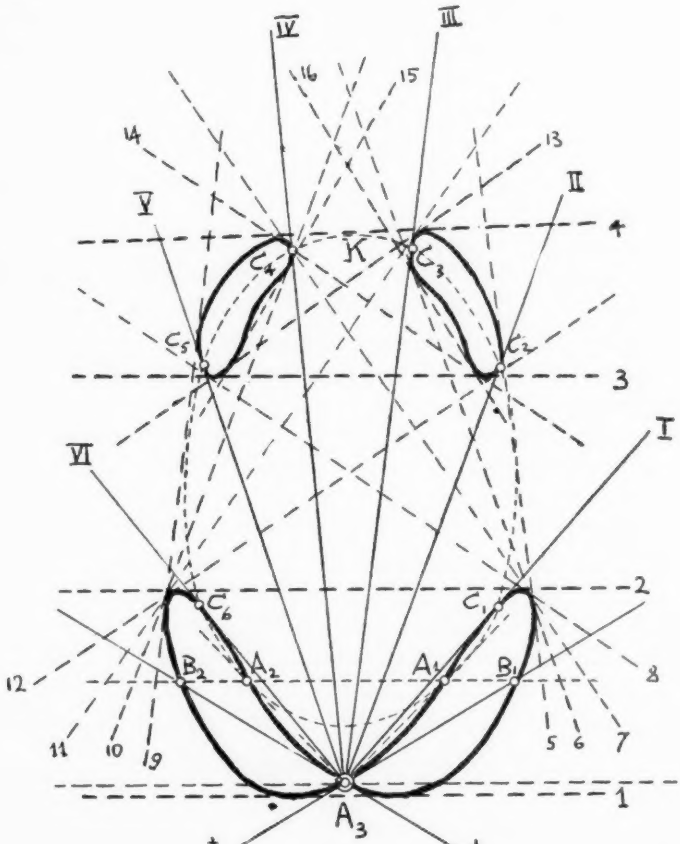
S. T. SANDERS.

The Plane Quartic of Genus Two

By ARNOLD EMCH
University of Illinois

1. *Introduction*

While the plane quartics of genus 3, 1, 0 have been rather extensively investigated, comparatively little has been done on the plane quartic of genus 2, the hyperelliptic plane quartic in its general aspect.



This is reflected in Gino Loria's excellent treatise on "Curve Piane Speciali Algebriche E Trascendenti", Vol. 1, pp. 119-128 (1930), in which very little attention is given to the quartic with genus $p=2$.

One of the most important properties of the hyperelliptic quartic was found by Bertini* as a particular case of hyperelliptic n-ics with an $(n-2)$-fold point as the only singularity: *From the only double point of the quartic six tangents (excepting the tangents to the two branches of the quartic at the double point) may be drawn whose contacts lie on a conic* (some call it the Bertini conic.)

Before that time the application of the theory of Theta and Abelian functions, as developed by Riemann, Weierstrass and others, to geometrical problems of a certain kind, demonstrated the power and illuminating character of these then new transcendental methods. In this field Clebsch did the pioneer-work, which has become classic.

G. Roch,† who studied under Riemann and became himself a very able and promising mathematician, applied this new theory to the hyperelliptic quartic and also to the general quartic.

Unfortunately the statements on the characteristics of the 16 double tangents and the 6 tangents from the double points are confused. It would appear that the latter are among the 16 effective double tangents.

P. Appell and E. Goursat give a very good account of some of the elementary applications of Abelian functions to geometric problems.‡ Although treating of the hyperelliptic quartic, the problem of the double tangents of such a quartic is ignored. As a matter of fact the statements made would make one believe that this problem is undetermined.

Among the most recent works we mention Enriques and Chisini's fourth volume of their well known "Lezioni".§ The geometric aspect of the theory is here shown in a very interesting fashion.

It is the purpose of this paper to give a short expository account of hyperelliptic plane quartics from the transcendental and algebro-geometric standpoint and to clear up some doubts which may arise when reading some of the papers mentioned above. *Moreover I shall for the first time give an actual example of such a curve when all 16 double tangents as well as the 6 tangents from the double point are real.* A freehand sketch of this possibility is shown in the dissertation‡ of

*Una nova proprieta delle curve di ordine n con un punto $(n-2)$ uplo. Atti della R. Accademia dei Lincei, Transunti, ser. 3, Vol. 1, pp. 92-95 (1876-1877.)

†Ueber die Doppeltangenten an Curven vierter Ordnung. Journal für die reine und angewandte Mathematik, Vol. 66, pp. 97-120. (Roch was born in 1839 and died 1866 in Venice, where he had gone for his health. He was Professor at the University of Halle).

‡Théorie des fonctions algébriques, pp. 499-503 (1895).

§Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche. Vol. IV, Funzioni Ellittiche e Abeliane.

‡On the forms of plane quartic curves. (Miss Gentry was instructor of mathematics at Vassar College.)

Ruth Gentry (1862-1917) at Bryn Mawr College in 1896. (Plate III, Fig. 1 (1)).

2. *The transcendental theory.*

1. Let C an hyperelliptic quartic in the plane (x,y) be given by the equation $f(x,y)=0$ with the double point at $(0,0)$ with two distinct tangents. The two Abelian integrals of the first kind attached to this curve are

$$(1) \qquad u=\int_{x_0,y_0}^{x,y}\frac{ax+by}{f'_y(x,y)}dx,\ v=\int_{x_0,y_0}^{x,y}\frac{cx+dy}{f'_y(x,y)}dx,\ ad-bc\neq 0.$$

The table of periods may be represented by

$$(2) \qquad \begin{vmatrix} \pi i & 0 & a_{11} & a_{12} \\ 0 & \pi i & a_{12} & a_{22} \end{vmatrix},$$

and the general periods of u and v by

$$(3) \qquad \begin{aligned} &\lambda\pi i+la_{11}+ma_{12} \text{ for } u,\\ &\mu\pi i+la_{12}+ma_{22} \text{ for } v, \end{aligned}$$

where λ, μ, l, m are integers. A generic line g cuts C in four points M_1, M_2, M_3, M_4 with the coordinates x_k, y_k; $k=1,2,3,4$. Taking these as the upper limits of the integrals u and v in (1) and denoting their values in these points by u_k and v_k, $k=1,2,3,4$, then according to Abel's theorem

$$(4) \qquad \begin{aligned} u_1+u_2+u_3+u_4&=P+\lambda\pi i+la_{11}+ma_{12}\\ v_1+v_2+v_3+v_4&=Q+\mu\pi i+la_{12}+ma_{22} \end{aligned}$$

in which P and Q are constants. When two points, say M_3 and M_4 are given, the first two M_1 and M_2 are uniquely determined and (4), according to the theorem of inversion, can be solved uniquely for u_1, u_2 and v_1, v_2. Now we impose the condition that g shall be a double tangent, so that the arguments in (4) by twos become equal, say $u_3=u_1$, $u_4=u_2$, $v_3=v_1$, $v_4=v_2$. This leads to the condition

$$(5) \qquad \begin{aligned} u_1+u_2&=\frac{P}{2}+\frac{\lambda\pi i+la_{11}+ma_{12}}{2},\\ v_1+v_2&=\frac{Q}{2}+\frac{\mu\pi i+la_{12}+ma_{22}}{2}, \end{aligned}$$

in which $\lambda, \mu, l, m \equiv 0, 1$ (mod. 2). This gives for the matrix $\begin{vmatrix} \lambda & \mu \\ l & m \end{vmatrix}$ sixteen solutions.

Theorem 1. *The hyperelliptic quartic has sixteen effective double tangents.*

But the situation is different when g passes through O, so that the pencil of g's cuts out the canonical series. The integrals u and v assume certain values at O which changes the values of the constants P and Q. Let P' and Q' be the new constants. The bisection of the canonical series $g_2{}^1$ gives the tangents from O to C different from the two at the branches of C at O. To the matrix $\begin{vmatrix} \lambda & \mu \\ l & m \end{vmatrix}$ we must apply the condition for odd characteristics: $\Sigma\lambda\mu \equiv 1$ (mod. 2). The number of solutions is, when p is the genus, $2^{p-1}(2^p-1)=2\cdot 3=6$.

Theorem 2. *From the double point of an hyperelliptic quartic there are six tangents to the curve (outside the two at O).*

In the continuous algebraic variation, depending upon a parameter, of a general quartic into one with a double point, it is readily seen that each of the six tangents from O to C absorbs two double tangents, so that together they count for 12 double tangents. Together with the 16 proper double tangents they complete the number of 28 as in the general case.

2. From the matrix $\begin{vmatrix} \lambda & \mu \\ l & m \end{vmatrix} \equiv 0,1$ (mod. 2) which determines the 16 double tangents choose any three, say

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}.$$

Adding corresponding indices we get a fourth matrix $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$. Thus any three of the 16 double tangents determine a fourth. We can form

$$\frac{16\cdot 15\cdot 14}{1\cdot 2\cdot 3}$$

such triples of double tangents with 6 points of contact. Through O draw a secant cutting C in two other points A and B. Through A,B,O and the 6 points of contact pass a cubic, which cuts C in two more points P_1 and P_2, which are the points of contact of the fourth double tangent determined by the triple. The fact is that all cubics on O, A, B determine a $g_8{}^6$ which contains the $g_8{}^5$ by all conics as an

incomplete series. Among these are the pairs of double tangents of the C. This is the reason why the above cubic cuts C in a fourth pair P_1, P_2 of points of contact of a double tangent. The four pairs of contacts can be chosen in four ways to form triples on the same cubic. Hence the

Theorem 3. *The 16 couples of contacts of the 16 double tangents of an hyperelliptic quartic lie*

$$\frac{16\cdot 15\cdot 4}{1\cdot 2\cdot 3}\cdot\frac{1}{4}=140$$

times by fours on cubics. Among these cubics are 60 degenerate ones each consisting of a conic and a line. It happens 60 times that the 8 points of contact of 4 double tangents lie on such a conic.

3. For the general quartic, the 56 points of contact of the 28 double tangents lie by eight on 315 conics, such that 4 points of contact of two double tangents lie on 5 other such conics. A pair of double tangents determines one of the 63 systems of four point contact conics enveloping the $C(p=3)$. In case of the $C(p=2)$ there are

$$\frac{16\cdot 15}{2}\cdot\frac{1}{6}=20$$

such systems. In each there are 6 pairs of double tangents which may be grouped into 15 couples. Consequently there are

$$\frac{20\cdot 15}{5}=60 \text{ conics,}$$

each containing the 8 points of contact of 4 proper double tangents, as stated in the theorem. In Enriques-Chisini this number is erroneously given as 75.

3. *The algebraic theory.*

1. In a projective $S_2(x_1x_2x_3)$ the quartic $C(p=2)$ may be written in the form

$$x_3^2\phi_2+x_3\phi_3+\phi_4=0 \tag{1}$$

where the ϕi are binary forms of degree i and x_1 and x_2. The tangents at the double point $A_3(001)$ are given by the two linear factors of $\phi_2=0$ and shall be denoted by t_1 and t_2. Let these cut C in B_1 and B_2. Join B_1 to B_2 by a line l, which we choose as $x_3=0$. Let l cut C in two

other points which may be chosen as $A_1(100)$ and $A_2(010)$; then C may be represented in the form

(2) $(x_3^2+x_1x_2)(ax_1^2+bx_1x_2+cx_2^2)+x_3(\alpha x_1^3+\beta x_1^2x_2+\gamma x_1x_2^2+\delta x_2^3)=0$

The first polar of A_3 with respect to C is now

(3) $$2x_3(ax_1^2+bx_1x_2+cx_2^2)+\alpha x_1^3+\beta x_1^2x_2+\gamma x_1x_2^2+\delta x_2^3=0$$

Multiplying (3) by x_3 and subtracting from the product thus obtained (2), we get

(4) $$(x_3^2-x_1x_2)(ax_1^2+bx_1x_2+cx_2^2)=0.$$

(3) cuts (2) outside of A_3 in the 6 points of tangency C_i, $i=1,\ldots,6$, of the tangents from A_3 to C.

Theorem 4. *As the intersections of* $ax_1^2+bx_1x_2+cx_2^2$ *with* C *do not coincide with* $C_1\ldots C_6$, *the latter lie on*

(5) $$x_3^2-x_1x_2=0$$

which is a conic K *touching* $x_1=0$, $x_2=0$ *at* A_2 *and* A_1.

Now draw any secant $s(x_2=\lambda x_1)$ through A_3. Setting $x_1/x_3=m$, s will cut (5) in two points determined by the parameters $m_1=1/\sqrt{\lambda}$, $m_2=-1/\sqrt{\lambda}$. The same secant cuts (2) outside of A_3 in two points whose parameters m' and m are the roots of

(6) $$m^2(c\lambda^3+b\lambda^2+a\lambda)+m(\delta\lambda^3+\gamma\lambda^2+\beta\lambda+\alpha)+c\lambda^2+b\lambda+a=0.$$

The four points with the parameters m', m, m_1, m_2 are harmonic when the cross-ratio $(m'\,m\,m_1\,m_2)=-1$, or

(7) $$2m'm-(m'+m)(m_1+m_2)+2m_1m_2=0$$

But $m_1+m_2=0$, $m_1m_2=-1/\lambda$. From (6) $m'm=1/\lambda$, hence (7) is identically satisfied. Hence the

Theorem 5. *The hyperelliptic quartic* C *is self-inverse with respect to the conical inversion with* $K=x_3^2-x_1x_2=0$ *as the conic and* A_3 *as the center of inversion* I. *The conic* K *cuts* C *in the 6 points of contact of the tangents from* A_3 *to* C. *When in* (2) $c=a$, $\delta=\alpha$, $\gamma=\beta$ *then* C *is also invariant in the quadratic transformation* $\rho x_1'=x_2x_3$, $\rho x_2'=x_3x_1$, $\rho x_3'=x_1x_2$.

2. The quartic C (2) gives rise to a DeJonquières transformation of order four. A secant s through A_3 cuts C outside of A_3 in two points P_1,P_2. On s choose any point P and determine P' on s such that $(P'P\,P_1P_2)=-1$. The couple P',P forms two corresponding points

in a DeJonquières transformation of order four. It has C as a pointwise invariant curve. A_3 is a fundamental point of order 2 with the polar of A_3 as the corresponding fundamental curve. This rational cubic with the same double point as C has, cuts C outside of A_3 in the six points of contact $C_1 \ldots C_6$ of the tangents to C from A_3. They are simple fundamental points with the tangents to C at these as fundamental lines. Explicitly the DeJonquières transformation has the form

$$\rho x_1' = x_1\{2x_3(ax_1^2+bx_1x_2+cx_2^2)+\alpha x_1^3+\beta x_1^2x_2+\gamma x_1x_2^2+\delta x_2^3\},$$

$$\rho x_2' = x_2\{2x_3(ax_1^2+bx_1x_2+cx_2^2)+\alpha x_1^3+\beta x_1^2x_2+\gamma x_1x_2^2+\delta x_2^3\},$$

$$\rho x_3' = x_3(\alpha x_1^3+\beta x_1^2x_2+\gamma x_1x_2^2+\delta x_2^3)+2x_1x_2(ax_1^2+bx_1x_2+cx_2^2).$$

It is easily verified that it leaves the conic K invariant. To the net of lines in S_2 corresponds the net of rational quartics with triple points at A_3 and through the $6C_i$. The study of invariant curves in DeJonquières transformations attached to hyperelliptic curves of order n with $(n-2)$-fold points at A_3 by Bertini led him to the discovery of the conic K as a notable case of many other classes of invariant curves, in the paper mentioned in the introduction.

4. *Example of an hyperelliptic quartic with 16 real double tangents and 6 real tangents from the double point.*

For the purpose of making a graph of an equation it is advisable and convenient to pass from the projective to the Cartesian plane (x,y). We make a coordinate transformation such that $x_1=0$ and $x_2=0$ go into $x_1=y-x$, $x_2=2(y+x)$, and $x_3=0$ into $x_3=(y-10)\sqrt{3}$. Moreover in formula (2) we choose $a=\frac{1}{2}$, $b=1$, $c=\frac{1}{8}$, and then imposing the condition of symmetry with respect to the y-axis, and choosing

$$y=\pm\frac{\sqrt{3}}{3}x$$

as tangents at the double point $A_3\equiv O$, the equation will assume the form

$$[3(y-10)^2-2(x^2-y^2)](3y^2-x^2)-(y-10)(px^2y+9y^3)=0.$$

In the figure the join of $B_1B_2A_1A_2$ is $y-10=0$. The unit of measurement is 2 millimeters. It takes considerable experimenting to obtain an equation which is represented approximately by the figure. This

happens when we choose for p and q, $p=5.95$, $q=-4.27$. The center of conical inversion is at O and the conic of inversion K is

$$3(y-10)^2+2(x^2-y^2)=0.$$

The tangents from O are lettered from I to VI, the double tangents from 1 to 16.

CORRIGENDA

On page 330 of volume XI the equation $r=\cos\,\Theta/2$ should be changed to $r=\cos^2\Theta/2$.

The 4th line from the bottom of page 217 should read:

$$(x^2-2x\sqrt{p}+p-q)(x^2+2x\sqrt{p}+p-t)=0$$

The 1st line at the top of page 218 should read,

$$C=2\cdot(t-q)\cdot\sqrt{p} \tag{2}$$

On page 218, center of the page, the 4th solution should be,

$$x_4=-\sqrt{p}-\sqrt{t}$$

and,

$$p^3-\frac{25}{2}p^2=\frac{769}{16}p-\frac{3600}{64}=0$$

should be replaced by

$$p^3-\frac{25}{2}p^2+\frac{769}{16}p-\frac{3600}{64}=0$$

A Study of the Angular Velocity About a Point Between the Foci in Keplerian Elliptic Motion

By M. WILES KELLER
Purdue University

Introduction. When a particle Q is moving in an elliptical path subject to a force at the focus R which varies according to the inverse square law, the angular velocity of Q about R is given by the well-known equation of areas

$$r^2 \frac{dv}{dt} = B. \tag{1}$$

In this equation r and v have their customary meaning and B is a constant.

To determine the point on the line joining the foci about which the angular velocity would be most uniform is an interesting problem that naturally suggests itself. Though this problem is relatively simple to state it presents some rather formidable difficulties when one attempts to solve it. Certain interesting facts can, however, be obtained with regard to the angular velocity about a point that is on the line joining the foci.

It will be shown that as the point takes on various positions between the foci there will sometimes exist several maxima and minima angular velocities. The number will depend upon the location of the point and the eccentricity of the ellipse. These critical values will be derived by a careful study of the cubic equation in two parameters that results when one seeks the values of v for which the angular velocity is a maximum and a minimum.

1. If any point P is chosen on the line joining the foci, the angular velocity about that point is given by the expression

$$\frac{d\Phi}{dt} = \frac{B[p+k(e+\cos v)]}{p[k^2+r^2+2kr\cos v]}, \tag{2}$$

where k is the distance from R to P, e is the eccentricity of the ellipse, Φ is the angle which the radius vector, r_1, from P to Q makes with the line joining the foci, and p is the semi latus rectum.

We see that when $k=0$, the equation for the angular velocity reduces to equation (1). The angular velocity of Q in this case steadily decreases as r increases, being greatest at the nearer apsis to R, and smallest at the farther.

Let us next study the case when $k=2ae$. Since $r_1=2a-r$ when $k=2ae$, we obtain at once from equation (2) that $d\Phi/dt=B/rr_1$. Therefore, we observe that the angular velocity of Q now is a minimum when $r=r_1$, and attains its largest value at the vertices.

2. We are thus led to seek the values of v for which $d\Phi/dt$ is a maximum and minimum for any specified value of k in the chosen interval $0\leqq k\leqq 2ae$.

To proceed, let $\psi=(P/B)\dfrac{d\Phi}{dt}$. Then, with values of r and dr/dv obtained from the polar equation of the ellipse, we have when collected in powers of $\cos v$

$$\frac{d\psi}{dv}=\frac{\left\{\begin{array}{l}-\sin v[k^2e^2(2p+ke)\cos^3 v+k^2e(2p+3ke)\cos^2 v\\ \quad +ke(3k^2-2kpe+p^2)\cos v+k^3-2k^2pe-kp^2+2kp^2e^2+2p^3e]\end{array}\right\}}{(k^2+r^2+2kr\cos v)^2(1+e\cos v)^3}$$

In order to determine the value of v which makes ψ a maximum and a minimum, we equate the preceding derivative to zero. It is seen at once that the denominator is always greater than zero. and that for any k the equation always has the roots $v=0$ and $v=\pi$.

To find the other roots we are led to the study of the cubic equation

$$\begin{aligned}(3)\qquad f(z)\equiv K^2e^2(2+Ke)z^3&+K^2e(2+3Ke)z^2\\ &+Ke(3K^2-2Ke+1)z+K^3-2K^2e-K+2Ke^2+2e=0,\end{aligned}$$

where p has been eliminated by the substitution $k=Kp$, and, for convenience, z has been put equal to $\cos v$. The substitution also gives $0\leqq K\leqq 2e/1-e^2=K_1$ as the interval for K equivalent to the given interval for k.

3. The cubic equation (3) is of the form

$$a_0(K,e)z^3+a_1(K,e)z^2+a_2(K,e)z+a_3(K,e)=0.$$

It is not difficult to show that the coefficients and the constant term are all positive when K and e are in their required intervals. Therefore, the cubic equation has no positive roots and the following theorem may be stated.

Theorem 1. For no position of the point P is there a point of minimum angular velocity as v assumes all values between $-\pi/2$ and $\pi/2$.

4. The smallest root which is pertinent in this discussion is $z=-1$. If this value of z is substituted in the cubic equation there results a cubic equation in K, $\bar{f}(K)$, which gives the values of K for which $z=-1$ is a root. Of the three roots of $\bar{f}(K)$,

$$K=K_2=\frac{-1+\sqrt{1+8e}}{2(1-e)}$$

is the only one in the required interval for K. From the factored form of $\bar{f}(K)$ it is easy to determine that $\bar{f}(K)>0$ for $0 \leqq K \leqq K_2$ and that $\bar{f}(K)<0$ for $K_1 \geqq K > K_2$. Also, from equation (3) it is seen that $f(0)>0$, and that $f(-\infty)<0$ for $0 \leqq K \leqq K_1$. Hence we have,

Theorem 2. For $K<K_2$ the cubic equation (3) *has two or no roots, and for $K>K_2$ three roots or one root between* -1 and 0.

If $K=K_2$ is substituted in equation (3) the roots of this equation may be located. The one root is, of course, $z=-1$. The other two roots are less than -1 when $0<e<2/9$, one less than -1 and one equal to -1 when $e=2/9$, and one less than -1 and one greater than -1 when $2/9<e<1$. This indicates that $K=K_2$, and $e=2/9$ are critical values of the cubic (3).

5. From the form of equation (3), it is seen that as K approaches zero the real root or roots become large negatively. This suggests that a K_3 can be found such that for $K \leqq K_3$ the roots of $f(z)=0$ are less than -1.

If the derivative of $f(z)$ is taken with regard to z and the discriminant of the resulting quadratic equation be restricted so that it is equal to or less than zero, then it is evident that the cubic equation has only one root. The value of K determined from the equality is

$$K=K_3=\frac{4+9e^2-\sqrt{16-72e^2+225e^4}}{12e(1-e^2)},$$

and it is less than K_2.

This K_3 is such a value that for $K \leqq K_3$ the cubic equation (3) has only one root. Consequently we may state

Theorem 3. If $K \leqq K_3$ the cubic (3) *has no roots for z between* -1 *and* 0.

It does not follow that for $K>K_3$ the cubic equation has roots in the interval -1 to 0. Such a K will be defined later.

TABLE A.

	Sign of $f(-1)$	*Interval for K*	*Sign of* Z_1	*Sign of* Z_2
Case 1. $1>e>2/5$	$-$	$K_2<K\gtreqless K_1$	$-$	$+$
	0	$K=K_2$	$-$	$+$
	$+$	$\frac{1}{3(1-e)}<K<K_2$	$-$	$+$
	$+$	$K=\frac{1}{3(1-e)}$	0	$+$
	$+$	$K_3<K<\frac{1}{3(1-e)}$	$+$	$+$
	$+$	$K=K_3$	$+$	$+$
Case 2. $e=2/5$	$-$	$K_2<K\gtreqless K_1$	$-$	$+$
	0	$K=K_2$	$-$	$+$
	$+$	$K_3<K<K_2$	$-$	$+$
	$+$	$K=K_3$	0	0
Case 3. $2/5>e>2/9$	$-$	$K_2<K\gtreqless K_1$	$-$	$+$
	0	$K=K_2$	$-$	$+$
	$+$	$\frac{1}{3(1-e)}<K<K_2$	$-$	$+$
	$+$	$K=\frac{1}{3(1-e)}$	$-$	0
	$+$	$K_3<K<\frac{1}{3(1-e)}$	$-$	$-$
	$+$	$K=K_3$	$-$	$-$
Case 4. $e=2/9$	$-$	$K_2<K\gtreqless K_1$	$-$	$+$
	0	$K=K_2$	$-$	0
	$+$	$K_3<K<K_2$	$-$	$-$
	$+$	$K=K_3$	$-$	$-$
Case 5. $2/9>e>1/5$	$-$	$\frac{1}{3(1-e)}<K\gtreqless K_1$	$-$	$+$
	$-$	$K=\frac{1}{3(1-e)}$	$-$	0
	$-$	$K_2<K<\frac{1}{3(1-e)}$	$-$	$-$
	0	$K=K_2$	$-$	$-$
	$+$	$K_3<K<K_2$	$-$	$-$
	$+$	$K=K_3$	$-$	$-$
Case 6. $1/5\geqq e>0$	$-$	$K_2<K\gtreqless K_1$	$-$	$-$
	0	$K=K_2$	$-$	$-$
	$+$	$K_3<K<K_2$	$-$	$-$
	$+$	$K=K_3$	$-$	$-$

6. In order to secure further information about the roots of equation (3), a study of the cubic transformed by the substitution $z = Z-1$ is necessary. The transformed equation is

$$\begin{aligned}(4)\quad f(Z-1) \equiv\, & K^2e^2(2+Ke)Z^3 + K^2e[3Ke(1-e)+2(1-3e)]Z^2 \\ & + Ke[3K(1-e)-1][K(1-e)-1]Z + (1-e)[K(1-e)-1][K(1-e) \\ & + \tfrac{1}{2}(1+\sqrt{1+8e})][K-K_2] = 0.\end{aligned}$$

From the derivative of this equation it may be determined whether the values of Z which make $f(Z-1)$ a maximum, Z_1, and a minimum, Z_2, are greater of less than zero for K and e in specified intervals. These facts, together with the sign of $f(-1)$ are given in Table A.

7. From the results obtained in the preceding sections we may now state the following facts about the roots of equation (3). They are:

(a) When $K_1 \geqq K > K_2$ and $0 < e < 1$ there is only one root.

(b) When $K = K_2$ and $2/9 \leqq e < 1$ there are two roots, $Z = -1$ and $0 > Z \geqq -1$, and when $0 < e < 2/9$ there is one root, $Z = -1$.

(c) When $K_2 > K \geqq 0$, and $0 < e \leqq 2/9$ there are no roots; the number of roots for the interval $2/9 < e < 1$ remains to be determined in section 8. For completeness we state these results. There are two roots for $\bar{K} \leqq K < K_2$ and no roots for $0 < K < \bar{K}$.

8. The preceding facts indicate that for any given e there is at least one value for K such that the z which makes $f(z)$ a minimum is also a double root of (3). It will be proved by a study of the discriminant of $f(z)$ that there is one and only one such value for K, which we shall designate by $\bar{K}$, where $K_2 \geqq \bar{K} > K_3$.

The discriminant is

$$(5)\qquad \Delta(K) = [-(1-e^2)K + 2e]\bar{\Delta}(K),$$

where

$$\begin{aligned}(6)\quad \bar{\Delta}(K) = 8(1-e^2)^2K^4 - 3e(1-e^2)(20-9e^2)K^3 - 2(4-3e^2)(1-18e^2)K^2 \\ + e(44-135e^2)K - 54e^2.\end{aligned}$$

If $f(z)$ has a double root only once as K varies from 0 to K_1, then equation (6) must have a single zero in that interval. It has been proved that the cubic (3) has, in the given interval of z, no roots for $0 \leqq K \leqq K_3$ and only one root for $K_2 < K \leqq K_1$. If, therefore, the cubic is to have one or more double roots for $0 < K \leqq K_1$, these K must be between K_3 and K_2. Because $\bar{\Delta}(K_1) > 0$ and $\bar{\Delta}(0) < 0$ we have that

the discriminant has at least one zero in the interval K_3 to K_2. It remains to be proved then, that there is only one zero in this interval.

Since at least one root of $\overline{\Delta}(K)=0$ is always less than zero, and since $\overline{\Delta}(K_1)>0$, we see that there will be three roots or one root of $\overline{\Delta}(K)=0$ less than K_1 and greater than zero. If three roots of $\overline{\Delta}(K)=0$ are less than K_1 and greater than zero, then $\overline{\Delta}'(K)=0$ (an accent, indicating the derivative of the function with regard to K) has all three roots less than K_1. It may be shown that $\overline{\Delta}'(K)=0$ never has three roots less than K_1 in the following manner. A value e_1 can be determined such that for $e<e_1$ the function $\overline{\Delta}(K)$ has its minimum for a $K>K_1$. When this is true $\overline{\Delta}(K)$ cannot vanish for three values of K less than K_1. Next a value $e_2<e_1$ can be obtained such that for $e>e_2$ we have $\overline{\Delta}'(K_1)<0$. Thus there will be a root of $\overline{\Delta}'(K)=0$ which is greater than K_1. Consequently, when $e \gtreqless e_1$ there cannot be three roots of $\overline{\Delta}'(K)=0$ less than K_1. It follows that for no value of e can three roots of $\overline{\Delta}(K)=0$ be less than K_1. Hence we may state:

Theorem 4. For $0 \leqq K \leqq K_1$ *and* $0<e<1$ *there is one and only one* K, $K=\overline{K}$, *for which the cubic equation* (3) *has a double root in the interval* $-1 \leqq z \leqq 0$ *and* $\overline{K}$ *is in the interval* K_3 *to* K_2.

It follows that when $2/9 \leqq e<1$, then $\overline{K}$ is such a value that for $K<\overline{K}$ the cubic equation (3) has no roots between -1 and 0. When $0<e \leqq 2/9$ then K_2 is such a value. These conclusions give immediately the result which we stated in section 7 for completeness.

9. By using the results obtained in the preceding sections concerning the roots of the cubic equation (3), it is possible to show graphically how the angular velocity appears to vary at a point P. Since an ellipse is symmetric, we shall only consider the angular velocity, $d\Phi/dt$, as v varies from 0 to π.

A maximum angular velocity is attained for $v=0$ so that the largest root of (3) will give a minimum, and the second largest a maximum when they exist.

The facts about the roots of (3) divide themselves into two distinct cases. The first case is where $0<e<2/9$, because for these values of e the double root which gives the cubic a minimum value is less than -1, and hence as K increases beyond K_2 a single root enters into the interval $-1 \leqq z \leqq 0$ through -1. The second case is where $2/9 \leqq e<1$, because the double root which makes the cubic a minimum suddenly enters in the interval $-1 \leqq z \leqq 0$ when $K=\overline{K}$ and these

roots gradually separate until for $K > K_2$ the one root has passed out of the given interval for z through -1.

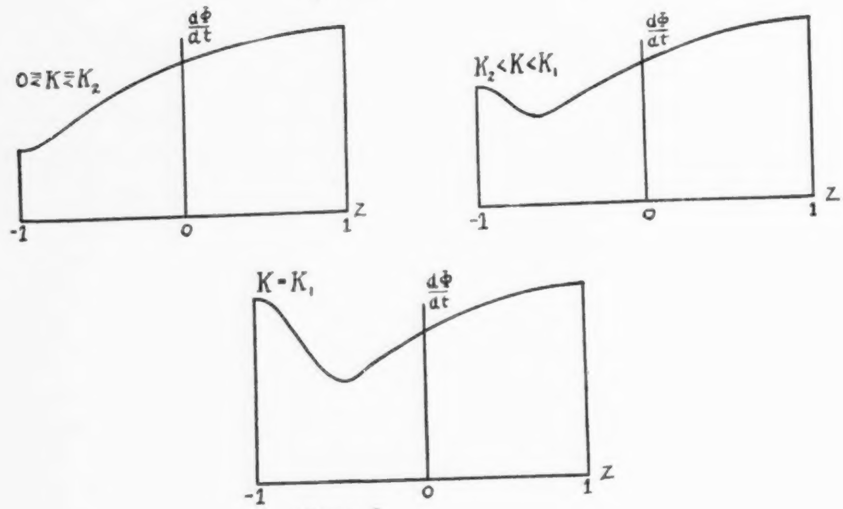
The graphs given in Table B indicate for a given e in the particular case how the angular velocity at any point P varies. Since the angular velocity is a continuous function of K, there will be a continuous deformation of the curve for a given e through the different stages indicated by the figure in each case as K increases.

For $0 \leqq K < \overline{K}$, and $K = K_1$ the graphs in Case 2 have the same relative shape as those in Case 1 for $0 \leqq K \leqq K_2$ and $K = K_1$ respectively. These graphs are, therefore, not repeated in Case 2.

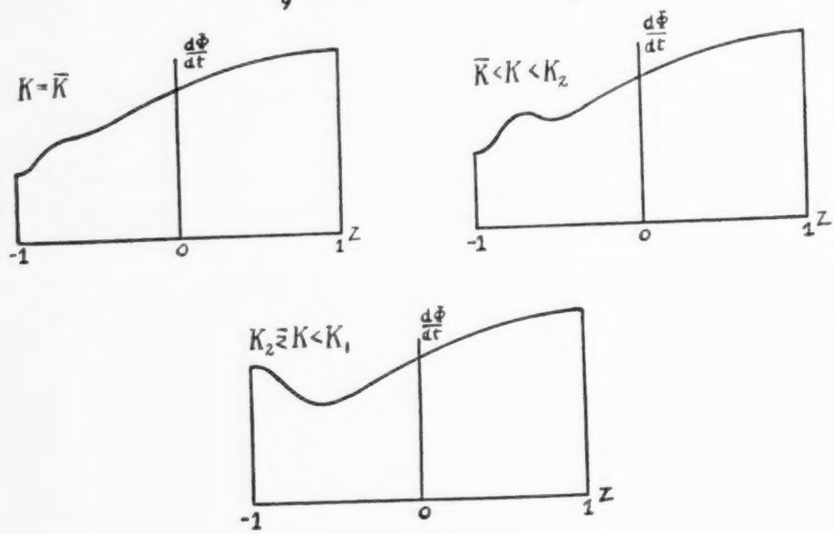
(See Table B page 20)

TABLE B

Case 1.
$\frac{2}{9} > e > 0$



Case 2.
$\frac{2}{9} \geqq e < 1$



The General Theory of Roulettes

By GORDON WALKER
Louisiana State University

The large class of curves classified as roulettes is an interesting type of locus problem. Familiar examples of roulettes are the cycloid, epicycloid, and the hypocycloid.

Definition: A *roulette* is defined to be the locus of

(a) a fixed point M, or (b) a variable point M, (such as the center of curvature), as the figure consisting of M and a mobile curve is allowed to roll along a stationary curve.*

A general theory of roulettes will be given for all curves such that the first derivative exists and is continuous and with the understanding that the point M is not necessarily on the mobile curve but that it continually retains the same relative position with respect to the mobile curve; or that the point M is some variable point such that corresponding to a point of the mobile curve,

$$P=(g(s),\, G(s)) \text{ there exists } M=(p(s),\, q(s))\dagger$$

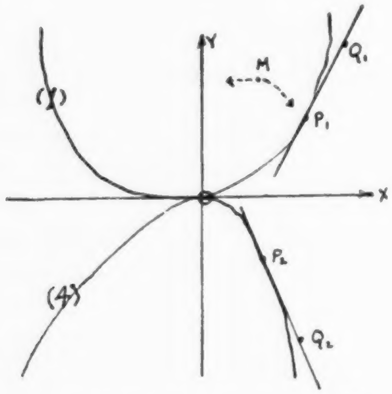
Let the mobile curve be:

$$(1)\quad \begin{cases} x=g(s), & \left(\dfrac{dg}{ds}\right)^2+\left(\dfrac{dG}{ds}\right)^2=1 \\[2ex] y=G(s), & g(o)=G(o)=\dfrac{d}{ds}G(o)=0, \qquad \dfrac{d}{ds}g(o)\neq 0 \end{cases}$$

Let $M=(p(s),\, q(s))$ be an arbitrary point. Now let the figure consisting of the mobile curve (1) and the point M be rolled along the stationary curve. We seek parametric equations of the locus of point M.

*The term *envelope-roulette* is applied to the envelope of a fixed line in the plane of the mobile curve.

†It was shown by Descartes that a line through M and the point of contact of the two curves is the normal to the roulette at M.



Take $P_1 = (g, G)$ so that the length of arc $\widehat{OP_1}$ will be

$$(2) \qquad \widehat{OP}_1 = \int_0^s \sqrt{\left(\frac{dg}{ds}\right)^2 + \left(\frac{dG}{ds}\right)^2}\, ds = s$$

and the tangent to (1) at P_1 will be

$$(3) \qquad \begin{cases} x = g + u\dfrac{dg}{ds} \\ y = G + u\dfrac{dG}{ds} \end{cases}$$

Take Q_1 on (3) so that $u = 1$, or

$$Q_1 = \left(g + \frac{dg}{ds}, \quad G + \frac{dG}{ds} \right)$$

and the distance $\overline{P_1Q_1}$ will be:

$$(4) \qquad \overline{P_1Q}_1 = \sqrt{\left(\frac{dg}{ds}\right)^2 + \left(\frac{dG}{ds}\right)^2} = 1$$

Now take as parametric equations of the stationary curve

$$(5) \qquad \begin{cases} x = f(s), \quad \left(\dfrac{df}{ds}\right)^2 + \left(\dfrac{dF}{ds}\right)^2 = 1 \\ y = F(s), \quad f(o) = F(o) = \dfrac{d}{ds}F(o) = 0, \qquad \dfrac{d}{ds}f(o) \neq 0 \end{cases}$$

Take $P_2=(f,F)$ so that the length of arc $\widehat{OP_2}$ will be

$$\widehat{OP_2}=\int_0^s\sqrt{\left(\frac{df}{ds}\right)^2+\left(\frac{dF}{ds}\right)^2}ds=s$$

and the tangent to (5) at P_2 will be

$$(6)\qquad \begin{cases} x=f+v\dfrac{df}{ds} \\ y=F+v\dfrac{dF}{ds} \end{cases}$$

Take Q_2 on (6) so that $v=1$, or

$$Q_2=\left(f+\frac{df}{ds},F+\frac{dF}{ds}\right)$$

and the distance $\overline{P_2Q_2}$ will be

$$\overline{P_2Q_2}=\sqrt{\left(\frac{df}{ds}\right)^2+\left(\frac{dF}{ds}\right)^2}=1$$

So that $\widehat{OP_1}=\widehat{OP_2}=s$ and $\overline{P_1Q_1}=\overline{P_2Q_2}=1$. Hence if the mobile curve (1) be rolled along the stationary curve (5) until P_1 becomes the point of mutual contact of the two curves, then P_1 will take the position P_2 which will bring the point Q_1 into the position Q_2. Let M' be the position taken by the point M at this time. We seek formulas for the coordinates of M'. To this end consider the Euclidean transformation:

$$(7)\qquad \begin{cases} x=h+a\bar{x}-b\bar{y} \\ y=k+b\bar{x}+a\bar{y} \qquad (a^2+b^2=1) \end{cases}$$

Let it carry P_1, Q_1 into P_2, Q_2 respectively. Then,

$$g=h+af-bF$$
$$G=k+bf+aF$$

$$g+\frac{dg}{ds}=h+a\left\{f+\frac{df}{ds}\right\}-b\left\{F+\frac{dF}{ds}\right\}$$

$$G+\frac{dG}{ds}=k+b\left\{f+\frac{df}{ds}\right\}+a\left\{F+\frac{dF}{ds}\right\}$$

Solving for a, b, h, k we have:

$$(8)\qquad \begin{cases} h=g-af+bF \\ k=G-bf-aF \\ a=\left(\frac{df}{ds}\right)\left(\frac{dg}{ds}\right)+\left(\frac{dF}{ds}\right)\left(\frac{dG}{ds}\right) \\ b=\left(\frac{df}{ds}\right)\left(\frac{dG}{ds}\right)-\left(\frac{dF}{ds}\right)\left(\frac{dg}{ds}\right) \end{cases}$$

Solving (7) for $\bar{x}$, $\bar{y}$ in terms of x, y, gives:

$$\begin{cases} \bar{x}=a(x-h)+b(y-k) \\ \bar{y}=-b(x-h)+a(y-k) \end{cases}$$

or by (8)

$$\begin{cases} \bar{x}=a(x-g)+b(y-G)+f \\ \bar{y}=-b(x-g)+a(y-G)+F \end{cases}$$

where a and b have the values also given in (8). But this transformation carries M into M'. Hence the locus of M is given by the equations:

$$(9)\qquad \begin{cases} x=\left\{\left(\frac{df}{ds}\right)\left(\frac{dg}{ds}\right)+\left(\frac{dF}{ds}\right)\left(\frac{dG}{ds}\right)\right\}(p-g) \\ \qquad\qquad +\left\{\left(\frac{df}{ds}\right)\left(\frac{dG}{ds}\right)-\left(\frac{dF}{ds}\right)\left(\frac{dg}{ds}\right)\right\}(q-G)+f \\ y=\left\{\left(\frac{dF}{ds}\right)\left(\frac{dg}{ds}\right)-\left(\frac{df}{ds}\right)\left(\frac{dG}{ds}\right)\right\}(p-g) \\ \qquad\qquad +\left\{\left(\frac{df}{ds}\right)\left(\frac{dg}{ds}\right)+\left(\frac{dF}{ds}\right)\left(\frac{dG}{ds}\right)\right\}(q-G)+F \end{cases}$$

Example and Special Case: To determine the locus of a point on a circle of radius r_1, as it is rolled on the outside of a circle of radius r_2. Let the mobile curve be:

$$\begin{cases} x = g(s) = r_1 \sin (s/r_1) \\ y = G(s) = r_1[1 - \cos (s/r_1)] \end{cases}$$

Let the stationary curve be:

$$\begin{cases} x = f(s) = r_2 \sin (s/r_2) \\ Y = F(s) = r_2 [\cos (s/r_2) - 1] \end{cases}$$

Now: $\dfrac{dg}{ds} = \cos (s/r_1)$, $\dfrac{dG}{ds} = \sin (s/r_1)$, $\dfrac{df}{ds} = \cos (s/r_2)$, $\dfrac{dF}{ds} = -\sin (s/r_2)$

and from (9)

$$x = \{\cos (1/r_1 + 1/r_2)s\}[p - r_1 \sin (s/r_1)] \\ + \{\sin(1/r + 1/r_2)s\}\{q - r_1[1 - \cos (s/r_1)]\} + r_2 \sin (s/r_2)$$

$$y = -\{\sin (1/r_1 + 1/r_2)s\}[p - r_1 \sin (s/r_1)] \\ + \{\cos(1/r_1 + 1/r_2)s\}\{q - r_1[1 - \cos (s/r_1)\} + r_2[\cos (s/r_2) - 1]$$

Take (p,q) as (o,o) and

$$x = (r_1 + r_2)\sin (s/r_2) - r_1 \sin(1/r_1 + 1/r_2)s$$

$$y = (r_1 + r_2)\cos (s/r_2) - r_1 \cos (1/r_1 + 1/r_2)s$$

which are the parametric equations of the epicycloid.

N.B. Any epicycloid can also be generated by a circle rolling with internal contact on the outside of a fixed circle. There is a similar double generation of every hypocycloid, a fact first noticed by Daniel Bernoulli (1725). The artist Albert Durer seems to have been the first (1525) to have considered a special case of an epicycloid.‡

In the 17th century La Hire, Desargues, Leibnitz, Réaumur, and Newton contributed to knowledge concerning these curves; among other things Newton showed (***Principia***) that all epicycloids and hypocloids are rectifiable.

The following interesting theorems on roulettes will be stated without proof as they follow by further applications of the general theory:

‡For many other beautiful forms of cyclic curves *see*, C. Taylor's *Curves Formed by the Action of . . . Geometric Chucks*, 2 Vols. (London, 1874, 1875.)

I. If any curve roll on an equal curve, corresponding points of the curves being in contact, the roulette of any point is a curve similar to a certain pedal of the fixed curve and of double its linear dimensions (Maclaurin, 1720).

II. Cayley's Sextic Curve is generated by the cusp of a cardoid rolling, with corresponding points in contact, on an equal cardoid. (Maclaurin).

III. The roulette of the vertex of a parabola rolling on a fixed equal parabola is a cissoid of Diocles.

IV. Every rhodonea can be generated as the roulette of a circle on a circle (Saurdi, 1752 and Ridolfi, 1844).

V. The hyperbolic spiral whose equation is $r=a/\theta$ rolled on a logarithimic curve $y=a \log (x/a)$ traces the axis of y or the asymptote. (Maxwell, 1849).

VI. The pole of a hyperbolic spiral rolling on a straight line traces out a tractix. (Demoulin, 1891).

VII. A helix rolls on a straight line to which it is always tangent while its axis moves in a plane; any point of the helix traces out a cycloid. (Besant, 1870).
N. B. A normal to the osculating plane of the cycloid will be the given straight line.

BIBLIOGRAPHY

W. H. Besant, *Notes on Roulettes and Glissettes,* (2nd ed., Cambridge, 1890).

J. C. Maxwell, *Scientific Papers,* Vol. 1, 1849.

R. C. Archibald, *Curves, Special,* Encyclopedia Britannica, 14th ed.

E. Wölffing's bibliography, *Bibl. Math.* (2), 1901.

R. A. Proctor, *Cycloid and....Cycloidal Curves,* (London, 1878).

T. S. Basley, *Epicycloidal Cutting Frame,* (London, 1872).

R. E. Mority, *Cyclic—Harmonic Curves,* (*Univ. of Wash. Pubs. in Math.,* 1923).

The Teachers' Department

Edited by
JOSEPH SEIDLIN and JAMES McGIFFERT

Vitalizing Mathematics*

By WILL E. EDINGTON
DePauw University

The past two decades have brought about a great change in attitude towards the teaching of mathematics in the public schools as well as towards the subject matter presented. Much adverse criticism has been given and much of it is really just, whether we mathematicians like it or not. The tremendous increase in public school enrollments, where it is said that 60 per cent of the children of high school age are actually in school, has brought its problems and among them such questions as the utility of the various subjects offered. Also the mass production method of teaching, which is generally followed, is certain to produce evil results in certain subjects, like mathematics, requiring sustained and consistent thinking. Many students fail, and this leads to dissatisfaction on the part of the tax paying public.

Some evidences of this dissatisfaction with the traditional school requirements of mathematics are the new courses of study adopted in New York and the experiments recently conducted in New England. The new course of study in New York provides that "only students particularly adapted to mathematics will be required to enter the subject's more abstruse realms." The courses offered "will strive to to be of practical value to the students in meeting real life economic problems". Dr. J. C. Morrison, assistant state commissioner for elementary education, is quoted as follows: "The new requirements shift the emphasis from skill in figures to an understanding of the number system employed in the economic world. In the last twelve years studies have been made to determine what changes in the social scale have occurred which have a direct bearing on the branches of mathematics taught under the present system. Changes in social conditions determine the social utility of mathematics."†

*Address of the retiring chairman of the Indiana Section of the Mathematical Association of America, May 1, 1937.

†Associated Press Report, February 6, 1937.

In an experiment in New England 5,000 school children have been studying fractions, half of the students studying the usual course and the other half studying only halves, thirds, fourths, eighths, twelfths and sixteenths. Dr. Guy M. Wilson, of Boston University, who is conducting the experiment, is quoted as follows: "The schools teach complicated and difficult arithmetic; the arithmetic outside of school is simple. So it must follow that the schools are teaching more arithmetic than is needed. Fractions form only 10 per cent of adult usage. The fraction $\frac{1}{2}$ alone makes up 60 per cent of all adult fractions. Halves with thirds and fourths make up 90 per cent. Few other fractions are needed, occasionally eighths, twelfths or sixteenths in special usage. Beyond this, reading knowledge only is needed."*

Now we mathematicians may not believe all this, and we may disagree with the disdom of dispensing with the teaching of much of our traditional mathematics, but every mathematician should be trained to work with and to recognize facts. One such fact is that the trend in schools even through the universities is away from mathematics. In a study of the trends during the five years from 1929 to 1935, made by Dr. Carl G. F. Franzen, of Indiana University, it is shown that the percentage of the student body engaged in the study of mathematics in 99 high schools, all members of the North Central Association, dropped from 57 per cent in 1929 to 49 per cent in 1935. Advanced arithmetic showed a loss of 33 per cent, whereas in general mathematics the increase was 112 per cent.

Of course we may argue that the educational pendulum at present is swinging towards the social sciences, but eventually it will swing back again. However, the swing has been so great in some universities as not only to cause embarrassment to the administration on account of the heavy loading in some departments with the corresponding light loading in others, but also to bring into question whether some of the universities shall be reduced to specialization rather than to the preparation of students along broad scholarly lines, as held forth as an ideal by the liberal arts colleges. The pendulum probably will swing back some, but to expect that the "good old days of mathematics" will return is to delude ourselves, and the sooner we mathematicians wake up to the facts and deal with them as we deal with the abstract facts in our mathematics, the sooner can we produce retardation of the pendulum in its swing away from mathematics and make our own social adjustments. Frankly we must adjust our courses honestly to meet the present social conditions. A great deal of the content of our courses now taught more or less as an end in itself

*United Press Report, December 29, 1935.

must be offered as a means. The emphasis must be placed not so much on the abstract, and we must teach with both hindsight and foresight. To do this we must vitalize our mathematics. This can be done, I believe, without sacrificing either the rigor or the beauty of mathematics, by humanizing it. In other words we must teach it with conviction and speak for it with authority. How can this be done? I shall offer the following suggestions which are consistent with the trend requiring the Bachelor's degree for the grade teacher and the Master's degree for the high school teacher. I say these suggestions are consistent with this trend for under these requirements there will be time for the beginning teachers of mathematics to be trained to humanize their teaching. This demands of the college teacher a sympathetic attitude toward fields other than his specialty, and an earnest effort on his part to humanize his special field by relating it wherever possible to the kindred fields and to practical life and experience. This will demand perspective of the past and present and a much broader outlook than is usually obtained in the major and minor method of college education.

I have classified my suggestions under five principal heads:

(1) Conscious, continual and consistent reference to the analogies between the mathematical processes and the ordinary processes met with in the work-a-day world.

(2) Dissemination of information on the application of mathematics in the various fields of human endeavor.

(3) Dissemination of knowledge of the parallel cultural development of mathematics with art, literature and science.

(4) A wider dissemination of certain foundational ideas of mathematics earlier in the student's training.

(5) The use of favorable propaganda based on fact and on the opinions of authorities and others who have recognized the values of mathematics as a cultural as well as a utilitarian subject.

Considering the first suggestion, the teacher of mathematics must recognize that in mathematics we do very little differently from what is done in every day life except that we make a careful analysis of our procedure and try to state explicitly our assumptions and definitions. One of the fundamental purposes of mathematics is to economize thought and to do it by thinking naturally, exactly and consistently. We are all aware of the analogies that may be made between mathematical processes and the ways of doing things in every-day life, but where many of us have failed in the past is in not making the

student conscious of these analogies by constantly and eternally impressing him with the analogies. It should be our purpose to convince the student that in working with mathematics he is working with something similar to what is used all the time by workers in the various fields of human endeavor. Once the student is convinced of this, the abstractions of mathematics will be accepted as a means and a result of careful analysis.

I shall point out some analogies which are very useful in putting across this idea. Thus x, the algebraic unknown, is indefinite like the abbreviations Dr. or St., and the real meaning or value may be determined only when restrictions or conditions, in other words, descriptive qualities, are given. x is similar to an unknown criminal to be run down by using certain known facts. Finding x is just like any other process where certain facts must be related and the resultant conclusions drawn. Thus x is no more abstract than John Doe in law, or the unknown sought for in chemistry, or a baffling illness diagnosed in a medical clinic. A system of coordinates is analogous to the plat of a well planned city, and in locating a point one is doing exactly the same as in locating a residence. Testing for a root by substituting in an equation is no different from testing with litmus paper in chemistry, testing soup for flavor by tasting, testing the roast for tenderness with a fork, or finding in a bunch of keys the particular key or keys that will fit a lock. To be successful demands a certain knowledge of the facts and a means for ascertaining the facts. Certain definitions must be known and certain assumptions must be made, whether the test is in mathematics or in every-day life. Transformation of coordinates may be compared to standardization in industry. Thus in analytic geometry we get rid of the xy-term in the equation of the conic in order to get the equation in a standard form whereby a more complete description of the conic is readily made. Almost all transformation in the integral calculus and differential equations is for the purpose of standardization whereby solutions may be more readily obtained. In automobile construction simplification is brought about by standardization. Certain formulas are known and the actual building of the automobile is made to conform to standardization in the fundamentals. Again substitituting in formulas is no different from the process of filling a prescription by the pharmacist, the baking of a cake by recipe, or unlocking a safe when one follows a formula of combination. The construction of an automobile or any other machine amounts to the substitution of mechanical devices for the variables in the different formulas. Determining the locus of a point under prescribed conditions is no different from hunting men with

certain qualifications to do a certain job. The constants and exponents of variables in equations which serve to identify certain characteristics of the curves are analogous to fingerprints, physical marks, and other features which might identify a human or a certain class of humans.

At first much of this may seem far fetched, but in my own classes, through the constant, conscious use of this analogy concept, I am firmly convinced that my students are beginning to look upon mathematics not as something abstract and distinctly different from other things they do, but as only a means of careful analysis of sets of conditions where sharp, careful thinking is required if trustworthy results are to be obtained. Thus the use of mathematics is natural to humans and not merely the tool of freaks. Once the student realizes that mathematics is a means of ascertaining truth no different in its fundamentals from determining the truth in every-day affairs he will look upon it as vital and something worth knowing and working for.

I shall now consider the second suggestion. In order to vitalize mathematics the teacher should have a broad, comprehensive knowledge of the applications of mathematics. He should have many specific references, if necessary, keep a card index of the literature, and be prepared to state the applications of certain processes to certain specific phenomena. Our textbooks do a certain amount of this in pointing out the shapes of crystals and snowflakes as geometrical patterns, and stating and illustrating the use of mathematics in engineering and business. However, the teacher of the future must go further than this, and it will be the business of the textbook writers to supply this need by citing applications in the many fields of human endeavor. Here the college instructor may also do an excellent piece of work in training the future teachers.

We are all aware of the applications of mathematics in engineering and the physical sciences, but probably few of us realize the extent to which mathematics is used in the natural sciences. Even in literature, how many of us who have read *Gulliver's Travels* are aware of the application of the theorems of similar surfaces and volumes in Swift's descriptions of the Lilliputians and Brobdingnagians? However, I shall confine my applications to the biological sciences. Much of what I give now is taken from a paper of mine which appeared in the *Proceedings of the Indiana Academy of Science* for 1926.

"Many of the laws well known to the mathematician, physicist and chemist have numerous applications in the natural sciences. Probably the best known is the compound interest law in which the form of growth is such that the rate of increase or decrease at any instant is proportional to the magnitude at that instant of that which

is increasing or decreasing. This law is met with in physics in the study of electric currents and the decrease of radio-activity, and in chemistry where it is known as Guldberg and Waage's law or the law of mass action. Thus $S_t = S_0 e^{-kv}$ where $v = t + (t^2/2p)$, where S_t is the area of a wound at time t, S_0 the initial area of the wound, and k and p constants, is the equation representing the healing of wounds. It has been used to study the merits of various antiseptics and dressings, and a serious deviation from the curve is often indicative of infection before the infection becomes apparent otherwise. If bacteria are allowed to grow freely in the presence of unlimited food their number at time t is given by the equation $N = Ce^{kt}$. This law is frequently met with in vegetable growth. The law of mass action is also applicable to the dissociation of oxyhaemoglobin. The growths of animal and human populations have been studied as applications of this law but owing to environmental, economic and other restrictive conditions several modified exponential equations have been used with better success to represent these growths." I once suggested the double use of this law in a student's thesis for the control of aphids in orchards by means of ladybugs.

"In his studies of the morphology of the blood vessels John Hunter found and stated that the angle of origin of the branch vessel varied in such a way that the circulation of the blood would be just sufficient for the part. Hess's law assumes that the loss of pressure is mainly due to the friction of the blood stream against the vessel walls and that the pressure varies directly as the length of the vessel and inversely as its radius, that is,

$$P = \left(\frac{M}{R} + \frac{N}{r}\right) K,$$

where M and N are the distances of the flow along the main vessel and branch vessel, respectively, and R and r are their respective radii, and K is the factor of proportionality. If now the minimum value of P be determined by means of the calculus, the following relation is found:

$$\frac{r}{R} = \cos x,$$

where x is the angle which the branch vessel makes with the main vessel. This relation shows that very small vessels like capillaries come off of larger vessels at practically right angles, branch vessels that are almost as large as the main vessel come off almost parallel

to the main vessel, and all vessels of practically the same size come off the same main vessel at the same angle."

"Again in the study of the nerves it has been found that if the outer coat or myelin sheath be considered as similar in its function to the insulating coat of a submarine cable, and that if the same law be assumed to hold for the speed of transmission of an impulse as for the speed of signalling along the cable, then the ratio between the radius of the axon or nerve core and that of the myelin sheath is such as to make the speed of the impulse approximately a maximum."

"There are numerous other special applications of mathematics to biology. Thus thermodynamical laws may be used in determining the amount of work done by the kidneys. The Van't Hoff-Arrhenius law for the influence of temperature on the velocity of reaction of substances in solution has applications in the conduction of impulses along a nerve, the rate of heart beat, the rhythm of the small intestine, respiration in plants, et cetera. The Schütz-Borissoff law with regard to the action of enzymes is applicable to gastric digestion in the early stages. The muscles fall into several distinct types and the work done by them can be approximately computed by mathematical methods. Still another interesting illustration is the relation between the tension and radius of curvature of a membrane under constant pressure. The thickness of wall is proportional to the tension which in turn is a function of the radius, and the variable thickness of the walls of the heart and uterus, for example, is easily explained by this relation."

"So far no mention has been made of Mendelian theory which is based on probabilities and requires a knowledge of the methods of mathematical theory of statistics for a proper understanding of it. The mathematical theory of statistics involves the use of calculus, some differential equations, the laws of probability, the theory of errors, et cetera, and may be applied wherever the number of variables is great and most of the variables are beyond control. It has been used heretofore chiefly in life insurance, educational and psychological tests, genetics and agricultural experimentation as a means of interpreting data and forecasting future possibilities."

The more advanced fields of mathematics are finding application in the natural sciences. Much is being done in the Departments of Physiology at Columbia, Johns Hopkins, Chicago universities and others. I shall cite two illustrations by quotations. Professor Horatio B. Williams, writing in *Science,** in 1929, stated "Our interest in this matter lies largely with the physicians of the coming generation.

**Science,* Vol. LXIX, May 17, 1929, pp. 505-509.

There was a time when the older men, not realizing their own meager knowledge of the fundamental sciences had proved in any way a handicap to them, advised prospective students of medicine not to waste time in extensive study of physics or mathematics. I think there is now less tendency to decry such preparation. When the speaker first came to Columbia some eighteen years ago, it was the exception to find a medical student who remembered anything of his physics with sufficient definiteness to make practical application of it. Gradually there has developed a tradition which has spread from the medical school to the premedical students that physics has its place in the foundations of the medical sciences and even in the practice of medicine. In recent years there has been an increasing number of students who have studied physics with interest and who retain such a grasp of the principles as to be able to make immediate application of them. I have also noted with interest the increasing number of medical students whose preparation in mathematics includes the calculus and the elements of differential equations. They find this preparation useful not alone in physics but also in chemistry, and from time to time they find further applications in physiology. I have recently received from Professor A. V. Hill in London, a reprint of a physiological paper on the diffusion of oxygen and lactic acid through tissues which is largely mathematical. Among the equations in this paper I noticed the Fourier equation of the

$$\frac{d^2u}{dx^2}+\frac{1}{x}\,\frac{du}{dx}+u=0,$$

the general solution of which is,

$$u=AJ_0(x)+BK_0(x),$$

where $J_0(x)$ is the Bessel's Function of zeroth order of the first kind and $K_0(x)$ is the Bessel's Function of zeroth order of the second kind."

"From this it will appear that the prospective student of medicine may carry his mathematical training rather farther than such students usually do and still find applications without going outside the field of medical sciences for them."

Consider the following quotation taken from an article by Kenneth S. Cole, of the Department of Physiology, Columbia University:

"The Fourier integral has proved to be a powerful and useful tool in many branches of science. In the Heaviside operational calculus form it has been particularly valuable in studying the transient behavior of electric circuits. When certain assumptions and approximations are made this type of analysis points out a relation between the

alternating current conductance and the direct current excitation of irritable biological tissues."

"Alternating current resistance and capacity measurements over a wide frequency range show that biologic materials may be considered electrically equivalent to a circuit containing two fixed resistances and a polarization element having an infinite impedance at zero frequency and a zero impedance at infinite frequency. This element may be considered as a resistance and a capacity in series both of which decrease with increasing frequency, n....... When a constant current i is started through this element at time $t=0$, the potential difference across the element may be found by either the Fourier integral or the operational method to be $e(t)=z_1\ it^{\alpha}/\Gamma(1+\alpha)$ where $\Gamma(1+\alpha)$m is the gamma function."†

Dr. Cole continues his discussion and its interpretation and results when applied to frog, toad and mammalian tissue.

Any one who wishes may find reports of many applications of mathematics in the biological sciences in such magazines as the *American Journal of Physiology, The Journal of General Physiology, The Journal of Experimental Medicine, and Biometrika.* Some of the leading writers along these lines are Raymond Pearl and A. J. Lotka, of Johns Hopkins, and P. Lecomte du Noüy, formerly of the Rockefeller Institute for Medical Research. Among the books along these lines the following will have special interest for their mathematical treatment: *Studies in Human Biology* and *Medical Biometry and Statistics* by Pearl, *Elements of Physical Biology* by Lotka, and a recent book *Biological Time,* by du Noüy, *Biomathematics* by W. M. Feldman and the well known *Growth and Form* by D'Arcy W. Thompson which contains a most interesting collection of illustrations of mathematical laws and curves found in plant and animal life.

The third suggestion for vitalizing mathematics consists in treating it at all times, wherever possible, in a manner similar to the study of literature. Many of the theorems and developments of mathematics are associated with the names of their discoverers, and with the rapidly increasing popularity of mathematical history, the opportunities for including some history with the usual subject matter are greatly increased.

One of the most fascinating ways to study the development of mathematics is to study briefly the contemporaneous development of other fields and to list the contemporary great leaders in the various fields. For example consider the period in history about 1500 A. D., say from 1450 to 1550. I shall mention only names of those who

*Science, Vol. 69, February 16, 1934, pp. 164-165.

lived and worked during that period without discussing their work, for I believe you will catch the spirit of what I am trying to present without a longer recital: Columbus, John Cabot, Balboa, Cortez, Magellan, Henry the Eighth, Ferdinand and Isabella, Ivan the Great, Thomas More, Erasmus, Rabelais, Tyndale, Raphael, Leonardo da Vinci, Titian, Andrea del Sarto, Botticelli, Albrecht Durer, Hans Holbein, Savanarola, Luther, Calvin, Copernicus, Widmann, Adam Reese, Ferro, Tartaglia, Cardan, Ferrari, Stifel, Recorde. What a tremendous perspective one has from this list of names. All represent originators of new ideas. Just as France, Spain and Russia were at this period really becoming nations, so we have pictured the real beginnings of mathematical symbolism and the solution of the cubic and quartic in algebra.

Take again a similar period around 1600: Queen Elizabeth, Philip II of Spain, Gustavus Adolphus, Richelieu, William, Prince of Orange, Hugo Grotius, Francis Drake, Walter Raleigh, Milton, Shakespeare, Spencer, Marlowe, Ben Johnson, Montaigne, Cervantes, Lope de Vega, Hobbes, van Dyck, Frans Hals, El Greco, Velasquez, Rubens, Rembrandt, Palestrina, Francis Bacon, Galileo, Kepler, Vieta, Napier, Desargues, Descartes, Cavalieri, Pascal.

Similar comparisons may be made for later periods, and as is to be expected the relationships become richer. The discovery and development of mathematics is the work of human beings, individuals who lived and worked and had their weaknesses and eccentricities, successes and failures. For some of us the study of mathematics for its own sake is sufficient, but for the great majority of our students it will be far more interesting if some human touches may be added. To study the development of mathematics in perspective and in comparison with the development in other fields is not to weaken our courses in mathematics. Is it not interesting to know that modern music began with Bach and Handel who lived in the same centuries as Newton and Leibnitz who began modern mathematics with the invention and the discoveries of the calculus? Might not both developments be traced to the cultural and economic developments of the preceding centuries?

Also supplementary material in the form of charts, maps, models, specimens of crystal forms and other geometrical forms from nature, pictures and lantern slides of past mathematicians, photographs and letters of living mathematicians, old textbooks, and certain illustrative apparatus, is always a valuable aid in vitalizing mathematics. Tuning forks, lenses and mirrors, et cetera, are all valuable in helping to make abstract ideas and figures more concrete.

The fourth suggestion, that of the use of certain ideas in mathematical courses earlier in the student's study of mathematics, is best illustrated in the concept of the negative number. Much of the student's difficulty in the understanding and use of negative numbers is due to the fact that the negative number is usually not completely nor properly defined. In taking up the study of the negative number the concept of the sense of direction is associated with the number as well as the concept of magnitude, and any attempt to avoid this fact leads only to confusion. There is no valid excuse for not defining and interpreting a negative number as a vector is defined, namely, as a quantity having both magnitude and direction which may be represented by a straight line of proper length in a certain direction. Then the negative number may be considered as derived from the positive number by rotating the vector representing the positive number through 180°. Then subtracting a negative number or multiplying a negative number by another negative number may be defined and interpreted as equivalent to a rotation of 180°, so far as the direction part of the process is concerned, so that the result is positive in direction. Thus the idea of the product of two negative numbers being positive appears at least reasonable to the student, for to him the idea of negative number as defined above becomes self evident from his past experience in reading and interpreting the thermometer, the ammeter on the automobile, and from discussions of debits and credits. The mystery and terror of $i=\sqrt{-1}$ may also be removed for the student by giving $i=\sqrt{-1}$, $i^2=-1$, an interpretation involving the vector concept in which multiplication by i represents a rotation of 90°, and i then fits in naturally with the ideas of positive and negative numbers. There need be no more confusion in thought about rotation than there is in confusing the temperature with the amount of heat in reading the thermometer. I shall develop this suggestion no further by examples on account of the lack of time, except to say that such a point of view replaces an abstract concept by a more concrete concept that appears reasonable and satisfying on account of its appeal to past experience.

The fifth suggestion is one in which the teacher points out through the statements of others the value of mathematics. If these statements come from non-mathematicians so much the better. For example, Bruce Barton, the well known author and business man, made the following statement several years ago in an editorial in *The American Magazine*: "When anyone asks me which of my college courses help me in the advertising business, I answer, 'Greek and mathematics'. Why? Because both these subjects compel the mind

to tackle a difficult problem and think it through to a conclusion." Exactly the same reason was offered this spring by the president of a large publishing house as his reason for desiring graduating mathematics majors to join his company with the idea of learning the business. The students in all of my classes have heard about both these statements for they make mathematics vital in fields that seemingly are independent of all mathematical preparation. I shall close by quoting one other fine example. Professor L. L. Thurstone, in *Science* for March 5, 1937, states, "A crucial matter in the development of a psychological science is the training program for the students who are to build this science. The first requisite is some familiarity with basic science, both physical and biological. Without this familiarity the student can hardly be expected to help in building a new science. If he expects to participate in making psychology a quantitative and a rational science, he must know something about the language of science, namely mathematics. The most profitable study of mathematics is probably done after it has become motivated by a realization that it does function, not merely as an aid or tool that a psychologist can use, but as the very language in which he thinks. It has been my experience that some students who are themselves unable to develop a mathematical idea are nevertheless well able to comprehend an essentially mathematical formulation of a psychological problem with its implications and experimental possiblities. Such a student may be more fertile with ideas than one who possesses considerable mathematical skill without the flexibility of mind that is essential in creative scientific work. More fortunate is the student who has all these aptitudes."

True indeed is the statement from the Talmud, "He who knows mathematics and does not make use of his knowledge, to him applies the verse in Isaiah (V, 12), 'They regard not the work of the Lord neither consider the operation of his hands'."

Mathematical World News

Edited by
L. J. ADAMS

Professor Frederick Wood, head of the department of mathematics at the University of Nevada, announces that Mr. Ingo Maddaus has been appointed instructor in mathematics.

Professor E. V. Huntington, Harvard University, delivered an invited address on *The Method of Postulates* at the meeting of the Institute of Philosophy of Bowdoin College held during the second week in April.

Professor J. L. Walsh submits the list of the appointments of the Division of Mathematics, Harvard University, for the academic year 1937-38; Benjamin Peirce instructors: Dr. H. M. MacNeille, Dr. A. E. Pitcher, Dr. Israel Halperin; part-time instructors: R. F. Clippinger, M. P. Fobes, A. D. Hestenes, R. F. Jackson, D. T. McClay, H. E. Robbins, A. Spitzbart.

The Wisconsin Section of the Mathematical Association of America met in Milwaukee on May 8, 1937. The following papers were presented:

1. *The computation of complex roots of algebraic equations with numerical coefficients.* Professor R. C. Huffer, Beloit College.

2. *A theorem on determinants.* Professor W. E. Roth, University of Wisconsin Extension Division, Milwaukee.

3. *The problem of Apollonius.* Mr. Stuart McNair, Senior High School, Sheboygan.

4. *Certain aspects of a teachers' training program as related to mathematics.* Dr. Paul Trump, Wisconsin High School, Madison.

Professor C. C. MacDuffee, University of Wisconsin, delivered an address on *Number Fields.*

The summer meetings of the American Mathematical Society and the Mathematical Association of America were held at the Pennsylvania State College on September 7-10, 1937. Professor John von Neumann, The Institute of Advanced Study, gave the Colloquium Lectures of the Society, on the subject *Continuous Geometry.* Professor Hassler Whitney, Harvard University, addressed the Society on *Topo-*

logical Properties of Differentiable Manifolds. Also, some fifty-two research papers were presented, and forty-six more were presented by title only.

Eighty-four Ph.D. degrees, with mathematics as the major, were awarded during the year 1936 by colleges and universities in the United States.

Professor Herbert Ellsworth Slaught, University of Chicago, died on May 21, 1937. Professor Slaught was a well known figure in mathematics, both here and abroad. At the time of his death he was Professor Emeritus at the University of Chicago and Honorary President of the Mathematical Association of America.

Professor R. D. James, University of California at Berkeley, gave a course in graduate mathematics entitled *The Viggo Brun Method in Number Theory* at the University of California at Los Angeles during the past summer session.

Professor W. M. Whyburn, head of the department of mathematics in the University of California at Los Angeles, gave a course in *Differential Equations* at the University of California at Berkeley during the past summer session.

Professor James McGiffert, Rensselaer Polytechnic Institute, delivered a radio address over KFWB, Los Angeles, on August 31, 1937. Professor McGiffert's subject was *A Trip Through Space on a Ray of Light.*

The eighteenth annual meeting of the Illinois section of the Mathematical Association of America, held at Northern Illinois State Teachers' College, De Kalb, Illinois, on May 14-15, 1937 included the following papers and addresses:

1. *Remarks Concerning Special Miquel Points.* Professor Gerald E. Moore, University of Illinois.

2. *The Removal of Certain Restrictions from Simpson's Rule.* Professor W. C. Krathwohl, Armour Institute of Technology.

3. *A Simplified Calculation of a Statistical Problem.* Professor Clifford N. Mills, Illinois State Normal University.

4. *Affine Differential Geometry of Curves and Ruled Surfaces.* Professor H. A. Simmons, Northwestern University.

5. *Explorations in New Dimensions.* Dr. Luise Lange, Woodrow Wilson Junior College, Chicago.

6. Illustrated lecture, *At the International Congress Last Summer.* Professor Rufus Oldenburger, Armour Institute of Technology.

7. *The Differential Equations of Certain Transversal Surfaces and their Transformations.* Professor G. D. Gore, Central Y. M. C. A. College, Chicago.

8. *Symposium on Mathematics in the Junior College.* Conducted by Dr. J. W. Peters, University of Illinois.

The Texas Section of the Mathematical Association of America met at the Rice Institute, Houston, Texas on April 23 and 24, 1937. The program was as follows:

1. *Roots of Polynomials.* Professor H. E. Bray, The Rice Institute.

2. *Polynomial Expansions in a Borel Region.* Dr. J. T. Hurt, A. & M. College of Texas.

3. *Simple Types of Semi-Rings.* Professor H. S. Vandiver, University of Texas.

4. *The Relations Between Solutions and Integrals of Systems of Differential Equations with Applications to the Three Body Problem.* Professor H. E. Buchanan, Tulane University.

5. *The Structure of Continua.* Professor R. L. Moore, University of Texas.

6. *On Surfaces of Negative Curvature.* Dr. E. F. Beckenbach, The Rice Institute.

7. *A New Approach to Euler Summability.* Mr. W. C. Mitchell, A. & M. College of Texas.

8. *Fuchsian Groups of Genus 2.* Mr. E. G. Kennedy, The Rice Institute.

9. *Mathematics as a Prescribed Subject for Graduation from College.* Professor J. N. Michie, Texas Technological College.

The Third National Convention of Kappa Mu Epsilon. The Convention was held at Mississippi State College on April 30 and May 1 with Professor C. D. Smith in charge of arrangements. All chapters were represented by official delegates and in many cases visiting delegates came so that more than one hundred came from ten different states. The meetings were presided over by Prof. J. A. G. Shirk of Kansas Alpha, National President. In addition to business and social meetings papers were presented by Dr. O. J. Peterson of Kansas Beta, Mr. R. A. Smith of Mississippi Beta, and Dr. Irby C.

Nichols of Louisiana Alpha. The following officers were elected to serve for the next two years.

President Pythagoras, Prof. J. A. G. Shirk, Kansas Alpha.

Vice-President Euclid, Prof. C. V. Newsom, New Mexico Alpha.

Secretary Diophantus, Miss E. Marie Hove, Nebraska Alpha.

Treasurer Newton, Prof. L. E. Pummill, Missouri Alpha.

Historian Hypatia, Miss Orpha Ann Culmer, Alabama Beta.

Reported by C. D. Smith, Retiring Secretary.

Mr. F. P. Welch is returning from leave after receiving the Ph.D. in mathematics at the University of Illinois. He has been promoted to the rank of Associate Professor of Mathematics at Mississippi State College.

Professor W. M. Whyburn, University of California at Los Angeles, announces the appointment of W. C. Risselman and F. A. Valentine as instructors and A. H. Diamond as lecturer in mathematics.

Occasional papers of significance to research workers in mathematics are found among the scientific papers of the Royal Society of South Africa. Professor W. A. Jolly is president of the Society, and its offices are located in the University of Cape Town, Rondebosch, Cape Town.

In addition to the well known mathematical journals of Japan there are several more elementary magazines published periodically by the various Japanese universities. Of these one of the more celebrated is that published by Tôkyô Butsuri Gakkô. The articles are all in Japanese, however. In each issue there is a bibliography of the contents of current issues of other mathematical periodicals from other countries.

Occasionally this department mentions foreign publishers of foreign mathematical works. To those already mentioned should be added the firms of Henry Sothern and Bernard Quatritch. Both of these firms are of London, England. They distribute free of charge very interesting and useful catalogs, completely annotated, and offer for sale standard and extremely rare mathematical works, ranging from familiar memoirs to an Italian illuminated manuscript, on vellum, of Euclid's *Elements*.

Problem Department

Edited by
ROBERT C. YATES

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to Robert C. Yates, College Park, Maryland.

SOLUTIONS

Late Solutions:

No. 137. *V. Thébault, Walter B. Clarke, E. R. Mann,* and *B. D. Mayo.* Thébault has done considerable work on this question in Education Mathématique, XXX, p. 105—L'Enseignement Mathématique (Geneve), 1934, pp. 349 to 359; 1935, pp. 309 to 324—Annales de la Societé Scientifique de Bruxelles, 1935, pp. 1 to 11.

No. 139. Given the triangle ABC. Find points P on AC and Q on BC such that $AP = PQ = QB$.

Solution* by *C. W. Trigg,* Cumnock College, Los Angeles.

On BC lay off $BD = AC$. Draw DM parallel to AB. With C as center and radius AC describe an arc cutting DM at E. Draw AE cutting BC at Q. Draw QP parallel to CE meeting AC at P. P and Q are the required points.

Proof: Through E draw a parallel to CB meeting AB produced at F. Then $EF = DB$, so $AC = CE = EF$. But the sides of the quadrilaterals $ACEF$ and $APQB$ are respectively parallel, so they are similar. Hence $AP = PQ = QB$.

Also solved by *J. W. Kitchens, James H. Duncan,* and *R. W. Douglas Clack.*

*A solution may be found in Altshiller-Court, "*College Geometry*", p. 44. See also *School Science and Mathematics*, 29, 644, (1929).

No. 140. Proposed by *Robert C. Yates*, University of Maryland.

Integrate: $\int \tan ax \cdot \tan bx \cdot \tan(a+b)x \cdot dx.$

Solution by *E. P. Starke*, Rutgers University.

The familiar formula $\tan(a+b)x = \dfrac{\tan ax + \tan bx}{1 - \tan ax \cdot \tan bx}$

reduces at once to

$$\tan ax \cdot \tan bx \cdot \tan(a+b)x = \tan(a+b)x - \tan ax - \tan bx.$$

The integration is now easy, and we have the result:

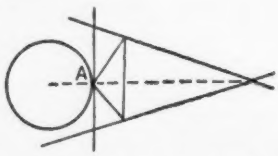
$$\frac{1}{a} \log \cos ax + \frac{1}{b} \log \cos bx - \frac{1}{a+b} \log \cos(a+b)x + C.$$

Also solved by *James H. Duncan*, *C. W. Trigg*, *William E. Byrne*, *Louis H. Kanter*, and *G. F. Aldrich*.

No. 141. Proposed by *Robert C. Yates*, University of Maryland.

Given two lines and a circle in the plane. Find that path-triangle of minimum perimeter whose vertices lie on these three elements.

Solution by the Proposer.



The required minimum path is that of a light ray which closes upon itself, – a triangle whose sides make equal angles with the given elements. Thus from the point of intersection of the two given lines a normal line is drawn to the given circle, meeting it in the point A. A tangent line is erected at A forming a triangle with the given lines. This tangent replaces the circle for our purposes and the solution is the well known pedal triangle.*

No. 142. Proposed by *Alfred Moessner*, Nurnberg-N, Germany.

What is the general solution of the identity:

$$xyz = x^2 + y^2 + z^2$$

in integers?

*See *Johnson's Modern Geometry*, pp. 253.

Solution by *E. P. Starke*, Rutgers University.

Given:

$$x^2+y^2+z^2=xyz. \tag{1}$$

Since no perfect square is congruent to 2 (mod 3), equation (1) has no solution unless x, y, and z are all multiples of 3.

If two letters are equal, say $x=y$, we have $2x^2+z^2=x^2z$ or

$$x^2=z^2/(z-2).$$

Thus x is an integer if and only if $z=3$ or 6 and $x=3$. Except then for the sets (3,3,3) and (3,3,6), every solution of (1) can be arranged with $x<y<z$.

Solving (1) for z, we may put the results:

$2z=xy+\sqrt{x^2y^2-4x^2-4y^2}$, $2z'=xy-\sqrt{x^2y^2-4x^2-4y^2}$, $z=xy-z'$. Note that $z'<y$, for if $3\leqq x<y$ we have $x^2<y^2(x-2)$ so that

$$x^2y^2-4xy^2+4y^2<x^2y^2-4x^2-4y^2$$

or

$$xy-2y<\sqrt{x^2y^2-4x^2-4y^2};$$

hence

$$xy-\sqrt{x^2y^2-4x^2-4y^2}<2y.$$

If (x,y,z) is a solution of (1) with $x<y<z$, we have immediately another solution (x_1,y_1,z_1) composed of the numbers x, y, and $z'=xy-z$ arranged in order of magnitude. Hence $z_1=y<z$; also, since y_1 is either x or z', $y_1<y$. Thus from any solution we derive a smaller one, unless $x=y=3$. Since there can be only a finite number of smaller solutions, we must arrive by successive reductions at the set (3,3,6) and finally (3,3,3).

But the process is reversible, and from each solution (x,y,z) we get two others, $(x,z,xz-y)$ and $(y,z,yz-x)$ unless $x=y$ (in which case there is only one new solution). We may therefore reach all solutions of (1) by starting with (3,3,3) to get (3,3,6). Then take (3,3,6) to get (3,6,15). From (3,6,15) get (3,15,39) and (6,15,87). From (3,15,39) get (3,39,102) and (15,39,582). And so on indefinitely.

It is also worth noting that:

(a) at least two of the numbers, x,y, and z are odd;

(b) no number is divisible by 4;

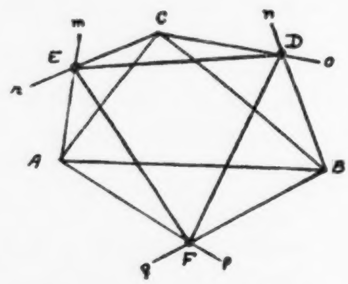
(c) except for the simple factors 3 and 2 (where it occurs), every factor of x,y, and z is of the form $4k+1$.

These all follow directly from the method of obtaining new solutions and, for (c), from the fact that, in $z=(x^2+y^2)/z'$, no factor of (x^2+y^2) can be of the form $4k-1$.

Also solved by *J. Rosenbaum* and partially so by *F. M. Kenny.* Rosenbaum remarks that "it should be determined whether or not it is possible to express all the solutions in terms of parameters, as is usually the case when the terms of the equation are all of the same degree. It would seem that when the terms are not of the same degree, as is the case here, then it should be impossible to express the solutions parametrically; especially when the equation is symmetrical in all the variables."

No. 143. Proposed by *Kirby W. Blain,* Lakeland, Florida.

Given any triangle ABC with lines m,n,o,p,q,r drawn at 30° angles with the sides as shown. Prove triangle DEF equilateral by elementary* methods.



Solution by *J. Rosenbaum,* Bloomfield, Conn.

The isosceles triangles BDC, CEA, AFB, have the sides a, b, c, of triangle ABC as bases and are similar. Hence the pairs of equal sides can be marked ra, rb, rc, respectively.

Let now ACP be an equilateral triangle, with the vertex P on the opposite side of AC from B, and let B,P be joined. In the triangles DCE and BCP the two sides DC, CE of the former are equal to ra and rb and the included angle is $60°+C$; the two sides BC, CP of the latter are equal to a and b, and the included angle, here too, is $60°+C$. Hence these triangles are similar, and therefore the third side, DE, of triangle DCE equals r times BP. In the same way, from the similarity of the triangles EAF and PAB, there is obtained that EF is also equal

*It is assumed here that the proposer desires a solution by the methods of Plane Geometry.

to r times BP; and hence DE equals EF. Since DE, EF is an arbitrary pair of sides of triangle DEF, it proves the triangle equilateral.

Also solved by *Walter B. Clarke, James H. Duncan, W. V. Parker,* and *C. W. Trigg.*

No. 144. Proposed by *R. A. Miller,* State University of Iowa.

Show that if the centroid of a triangle lies on the inscribed circle, then $5(\sin^2 A+\sin^2 B+\sin^2 C)=6(\sin A\sin B+\sin A\sin C+\sin B\sin C)$.

Solution by *C. W. Trigg,* Cumnock College, Los Angeles.

In the triangle ABC let M be the median to side b, and x the segment of the median internal to the incircle. The distance of A from the point of contact of the incircle with b is equal to $(s-a)$, where $s=(a+b+c)/2$. Since a tangent to a circle is the mean proportional between the entire secant and its external segment.

$$(s-a-b/2)^2=m(m/3+x)/3$$

and
$$(s-b)^2=2m(2m/3-x)/3.$$

Now
$$a^2+c^2=2m^2+b^2/2,$$

so m, x, and s may be eliminated from these equalities, yielding

$$5(a^2+b^2+c^2)=6(ab+ac+bc).$$

When this equation is multiplied through by k^2, where k is the ratio between the sine of any angle and the opposite side, the desired result is obtained.

Also solved by *C. E. Springer, T. Mahrenholz,* and the Proposer.

Editor's Note. Both Springer and Mahrenholz make use of the equation of the incircle in areal coordinates:

$$\sqrt{(s-a)x}+\sqrt{(s-b)y}+\sqrt{(s-c)z}=0.$$

The centroid (1/3,1/3,1/3) is on this circle if:

$$\sqrt{(s-a)}+\sqrt{(s-b)}+\sqrt{(s-c)}=0,$$

which reduces, on removing radicals, to the desired expression.

Mahrenholz lists the following properties of the triangle:

(a) The Nagel point lies on the Steiner ellipse inscribed to the triangle. (Mathesis, 1923---162).

(b) $s^2 = r(4R+r) = r(r_a+r_b+r_c)$.

$s/r = \tan A/2 + \tan B/2 + \tan C/2$.

$5 \cdot \cot \omega = 6R(1/a+1/b+1/c)$, where ω is the Brocard angle.

(Mathesis, 1931—382).

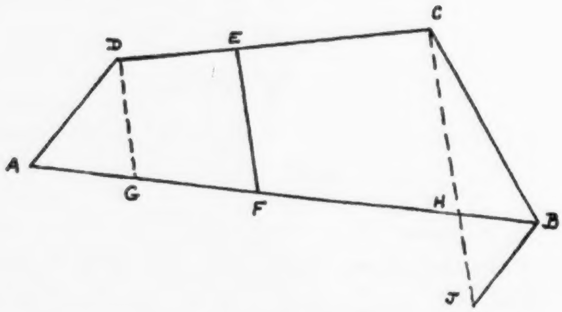
See also Mathesis, 1924—314 and 407.

No. 145. Proposed by *E. H. Umberger*, University of Maryland.

Given the quadrilateral $ABCD$ with the points E, F dividing the opposite sides DC and AB, respectively, in the ratio $\lambda : \mu$, where $\lambda + \mu = 1$; $\lambda, \mu > 0$. Prove

$$EF \leqq \lambda \cdot BC + \mu \cdot AD.$$

Solution by the Proposer.



Draw DG and CH parallel to EF, and cutting AB, or AB produced in G and H. Draw BJ through B parallel to AD and cutting CH produced in J. We have:

$$DE/CE = AF/BF = FG/FH = AG/BH = DG/HJ = AD/BJ = \lambda/\mu.$$

Since EF divides the non-parallel sides of the trapezoid $CDGH$ in the given ratio, we have also:

$$EF = \lambda \cdot CH + \mu \cdot DG = \lambda \cdot (CH + HJ).$$

Now $$CH + HJ \leqq BC + BJ,$$

and hence $$EF/\lambda \leqq BC + (\mu/\lambda) \cdot AD$$

or $$EF \leqq \lambda \cdot BC + \mu \cdot AD.$$

Also solved by *J. Rosenbaum*.

No. 147. Proposed by *A. Moessner*, Nurnberg-N, Germany.

What is the general solution of the diophantine system:

$$\left.\begin{aligned} 2x^5 &= a^4+b^4+c^4 \\ 2x^{10} &= a^8+b^8+c^8 \end{aligned}\right\} ?$$

Solution by *T. Mahrenholz*, Gottbus, Germany.

Suppose that

$$c^2 = a^2+b^2. \tag{1}$$

Then
$$\begin{aligned} 2x^{10} &= 2(a^8+2a^6b^2+3a^4b^4+2a^2b^6+b^8) \\ &= 2(a^4+a^2b^2+b^4)^2, \end{aligned}$$

and the solution of the problem is found by taking

$$x^5 = a^4+a^2b^2+b^4.$$

By relation (1), a and b may be written in terms of parameters α, β:*

$$\left.\begin{aligned} a &= n(\alpha^2-\beta^2), \\ b &= 2n\cdot\alpha\beta. \end{aligned}\right\} \tag{2}$$

Hence $$x^5 = n^4(\alpha^8+14\alpha^4\beta^4+\beta^8).$$

Now take $$n = \alpha^8+14\alpha^4\beta^4+\beta^8,$$

then $$x = \alpha^8+14\alpha^4\beta^4+\beta^8.$$

Substituting in (2) and (1) we obtain the solutions:

$$a = (\alpha^2-\beta^2)(\alpha^8+14\alpha^4\beta^4+\beta^8)$$

$$b = 2\alpha\beta(\alpha^8+14\alpha^4\beta^4+\beta^8)$$

$$c = (\alpha^2+\beta^2)(\alpha^8+14\alpha^4\beta^4+\beta^8).$$

For certain small values of α and β, particular solutions are:

α	β	x	a	b	c
2	1	481	1443	1924	2405
3	2	24961	124805	299532	324493
4	1	69421	552968	1036815	1175057
4	3	362401	2536807	8697624	9060025
5	2	530881	10617620	11148501	15395549
5	4	2696161	24265449	107846440	110542601

*See L. E. Dickson, *History of the Theory of Numbers*, Vol. II, Chapter 4.—Ed.

Editor's Note: Since (1) is not shown to be a necessary result of the original system, the above solution is not proved unique; i. e., it is not certain that all values of α, β yield all solutions of the system.

No. 148. Proposed by *V. Thébault*, Le Mans, France.

With the digits 0,1,2,3,4,5,6,7,8,9, taken twice, form two numbers which are respectively the square and the cube of the same number.

Solution by *C. W. Trigg*, Cumnock College, Los Angeles.

$$4641 < N < 10{,}000,$$

since only numbers within these limits have twenty digits in N^2+N^3. Since N^2+N^3 is divisible by 9,* it follows that N is either a multiple of 3 or else one less than a multiple of 9. Now N^2 and N^3 together must contain each digit exactly twice, so N cannot end in 0. Under these restrictions (and most conveniently with the aid of a table of the squares and cubes of numbers up to 10,000, e. g. Barlow's) we may then identify the unique solution,

$$(6534)^2 = 42693156,$$

$$(6534)^3 = 278957081304.$$

Also solved by the Proposer. *T. Mahrenholz* solved a similar problem in which the digits were used only once. See *American Mathematical Monthly*, 42, p. 175, March 1935.

No. 149. Proposed by *V. Thébault*, Le Mans, France.

Find a perfect square of ten digits of the form:

$$aabbccddee.$$

Solution by the Proposer.

We have evidently,

$$e = 0 \text{ or } 4,\dagger$$

and it is easily shown that:

$$a+b+c+d+e = 0, \pmod{11}$$

$$aa+bb+cc+dd+ee = 0, \pmod{11}\ddagger$$

*The sum of the digits in N^2+N^3: $0+1+\ldots.+9+0+1+\ldots.+9=90$.—Ed.

†Which may be verified by trial.

‡A number $N = a_k\ldots\ldots a_3a_2a_1a_0$ may be written as $a_0+10a_1+100a_2+\ldots\ldots+10^ka_k$ and as $a_0-a_1+a_2-a_3+\ldots.+11a_1+99a_2+1001a_3+9999a_4+\ldots.$ Thus N is divisible by 11 if $a_0-a_1+a_2-a_3+\ldots.$ is divisible by 11. Here, obviously, *aabbccddee* is divisible by 11, the quotient being *a0b0c0d0e*, and thus $a+b+c+d+e$ is divisible by 11. The second expression may be written as $11(a+b+c+d+e)$ and is therefore zero mod 11^2.—Ed.

It follows that:

$$a+b+c+d+e=11$$

$$a+b+c+d=11 \text{ or } 7 \qquad \text{since } e=0 \text{ or } 4.$$

There are two solutions then:

$$(88000)^2=7744000000$$

$$(74162)^2=5500002244.$$

Also solved by *T. Mahrenholz* and *C. W. Trigg.*

No. 151. Proposed by *A. C. Briggs,* Wilmington, Ohio.

A regular hexagon is composed of jointed rods, each of weight W. It is placed in a vertical plane with one rod resting on a table. Its right- and left-hand vertices are connected by a horizontal string of negligible mass. Find the tension in this string if a weight M is placed at the mid-point of the top rod.

Solution by *C. E. Springer,* University of Oklahoma.

Let $4s$ be the distance between the horizontal bars, and $2y$ the length of the string. Then if the system be given a slight displacement, the top rod will descend a distance $4\delta s$, the next two lower bars will descend a distance $3\delta s$, and the next two bars a distance δs. The lowest rod is stationary. By the principle of virtual work:

$$(M+W)(4\delta s)+2W(3\delta s)+2W(\delta s)+T(2\delta y)=0.$$

Now $s=a\cdot\sin\Theta$, where $2a$ is the length of a rod, and Θ is the angle between an inclined rod and the string, and $y=a+2a\cdot\cos\Theta$, so that

$$\delta s=a\cdot\cos\Theta\delta\Theta=(a/2)\delta\Theta$$

and

$$\delta y=-2a\cdot\sin\Theta\delta\Theta=-a\sqrt{3}\cdot\delta\Theta.$$

The above equation of virtual work then gives:

$$T=(1/\sqrt{3})(M+3W).$$

No. 152. Proposed by *R. E. Gaines,* University of Richmond, Virginia.

If OA_1, OA_2, OA_3 be the semi-diameters drawn through the mid-points, M_1, M_2, M_3, of the sides of a triangle inscribed in an ellipse, and if the ratios OM_1: OA_1, etc., be denoted by r_1, r_2, r_3, then

$$r_1^2+r_2^2+r_3^2\pm 2r_1r_2r_3=1,$$

the double sign being plus or minus according as the center of the ellipse is inside or outside the triangle.

Solution by *C. E. Springer*, University of Oklahoma.

Let the eccentric angles of the vertices of the triangle be α, β, γ. Then the mid-point M_1, has coordinates:

$$\left[a \cos\frac{\beta+\gamma}{2} \cos\frac{\beta-\gamma}{2},\ b \sin\frac{\beta+\gamma}{2} \cos\frac{\beta-\gamma}{2}\right]$$

and the line OM_1 intersects the ellipse in the point A_1, the x-coordinate of which is

$$a \cdot \cos \frac{\beta+\gamma}{2}.$$

Then $$r_1 = OM_1/OA_1 = \cos \frac{\beta-\gamma}{2}.$$

Similarly, $$r_2 = \cos \frac{\gamma-\alpha}{2} \text{ and } r_3 = \cos \frac{\alpha-\beta}{2}.$$

It is an exercise in trigonometry to show that

$$\sum \cos^2 \frac{\beta-\gamma}{2} = 1 \pm 2\pi \cos \frac{\beta-\gamma}{2},$$

since $$\sum \frac{\beta-\gamma}{2} = 0$$

The negative sign occurs only if one of the three angles is obtuse, in which case the center of the ellipse is inside the triangle. Hence, we have

$$r_1^2 + r_2^2 + r_3^2 \pm 2r_1r_2r_3 = 1,$$

where the positive sign is taken if the center of the ellipse is inside the triangle.

PROPOSALS

No. 157. Proposed by *B. D. Mayo*, Virginia Military Institute, Lexington.

The distances of the three nearest vertices of a square from a given fixed point are a,b,c, where $a \neq b \neq c$. Construct the square.

No. 158. Proposed by *William E. Byrne*, Virginia Military Institute, Lexington.

In many texts the following problem appears:
"Prove the identity

$$2\cdot\text{arc tan } x = \text{arc sin } 2x/(1+x^2)."$$

What are the precise relations between the two functions of x when principal values of the angles are used? That is, when

$$-\pi/2 < \text{arc tan } x < \pi/2 \text{ and } -\pi/2 \leqq \text{arc sin } x \leqq \pi/2.$$

No. 159. Proposed by *T. Dantzig*, University of Maryland, College Park.

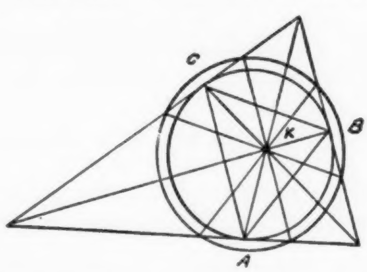
Two planes, α, β, make an angle Θ with each other. A triangle in α is in perspective from a point P with another triangle in β. Show that as Θ varies P describes a circle.

No. 160. Proposed by *E. P. Starke*, Rutgers University.

Show that the sum of the exponents of the highest power of 2 not greater than $2n$, $2n/3$, $2n/5, \ldots$, respectively, equals n. Can this be generalized?

No. 161. Proposed by *Frank Morley*, Baltimore, Maryland.

It is known that parallels to the sides of a triangle ABC through the symmedian point K meet the sides on a circle. Prove that these



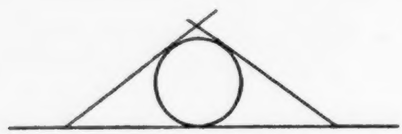
parallels meet the tangents at A, B, C, to the circumcircle in points on a circle.

No. 162. Proposed by *Walter B. Clarke*, San Jose, California.

Three cevians AD, BE, CF of a given triangle ABC determine the point P. New cevians are formed by interchanging the segments of each side, these determining the point P'. How is P to be located so that the distance from centroid to PP' shall be maximum?

No. 163. Proposed by *Walter B. Clarke*, San Jose, California.

Two rigid rods of indefinite length are hinged to a straight track at points 6 feet 2 inches apart. A wheel 2 feet in diameter is placed midway between the rods and rolled along the track with the rods



constantly touching it. (a) How far must the wheel roll until the rods are parallel? (b) What is the locus of the intersection of the rods?

No. 164. Proposed by *V. Thébault*, Le Mans, France.

A line L cuts the sides BC, CA, AB of a triangle T in the points α, β, γ. Show that

(1) The parallels to BC, CA, AB through the centers of the circles circumscribed upon $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ determine a triangle T' symmetrically equal to T.

(2) The center of symmetry of the triangles T and T' is situated on the Newton line of the quadrilateral $ABCL$.

No. 165. Proposed by *V. Thébault*, Le Mans, France.

With the digits 0,1,2,3,4,5,6,7,8,9, taken once, form five numbers such that they may be arranged in an arithmetic progression.

No. 166. Proposed by *Karleton W. Crain*, Purdue University.

Find and describe the envelope of the Euler line of the triangle ABC, where B and C are fixed and A moves along the line AC.

No. 167. Proposed by *Alfred Moessner*, Nurnberg-N, Germany.

What is the general solution of the system:

$$A+B+C+D=E+F+G$$

$$A^3+B^3+C^3+D^3=E^3+F^3+G^3$$

$$A^4+B^4+C^4+D^4=E^4+F^4+G^4$$

in integers?

No. 168. Proposed by *H. D. Grossman*, New York City.

N coins are tossed and all that fall heads are removed. The remaining ones are tossed again after which all heads are again removed. How many times must this operation be repeated if the probability that no coins remain is to be at least 1/2?

No. 169. Proposed by *V. Thébault*, Le Mans, France.

Upon the sides AB, BC, CD, DA of a parallelogram $ABCD$ construct externally and internally the squares whose centers are E, F, G, H and E', F', G', H'.

(1) The quadrilaterals $EFGH$ and $E'F'G'H'$ are the squares the sum of whose areas is equal to that of the squares constructed on two consecutive sides of the parallelogram.

(2) Construct the parallelogram $ABCD$ knowing the centers E, F, G, H, (or E', F', G', H') of the squares constructed upon its sides.

(3) Determine the parallelogram $ABCD$ if each side of the square $E'F'G'H'$ passes through one of the vertices of the square $EFGH$.

Reviews and Abstracts

Edited by
P. K. SMITH

Higher Algebra. By S. Barnard and J. M. Child. Macmillan and Company, London, 1936. xiv+585 pages.

This is the first volume of a two volume series by these authors. It is not just another "college algebra" like so many in this country. It will serve better as a reference book for several courses as given in the American schools than as a text for any one course.

According to the preface "The authors have aimed at producing a treatise in which the subject is developed logically, complete so far as it goes and serving as an introduction to modern analysis." It seems to the reviewer that they have accomplished this aim to a reasonable degree.

Much of the more advanced material found in the usual college algebra is included. In addition to this the book contains sufficient material for a first course in the theory of equations and also a first course in the theory of numbers. Limits and continuity are discussed and the rules of differentiation are proved. Infinite series and products are discussed rather fully and the usual tests for convergence are given. In the chapter on determinants the usual elementary theory is developed and special attention is given to the symmetric and skew-symmetric types.

The book as a whole is very well written. The paper and type are good with comparatively few errors. Even though this book may not prove suitable as a text for courses in the American schools, it will appeal to a large number of teachers because of the many exercises to be found in it. There are about fifteen hundred exercises with varying degrees of difficulty and interest. Some of these have been gathered from other books but many are original.

Louisiana State University. W. V. PARKER.

Plane Trigonometry. By H. A. Simmons and G. D. Gore. John Wiley and Sons, New York, 1937. viii+201 pages+66 pages of tables.

In the preface of this text is found the statement: "The authors' purpose in writing this book is to give in simple language and with

suitable geometric illustration the best available proof of each type used, to offer a large and diversified set of appropriate problems possessing considerable newness, to alternate exposition and problems in such a way as to make the work as simple and convenient as possible both for the student and teachers, and to use tables which are extremely convenient."

This text is justified on the ground of its simplicity and at the same time without the sacrifice of a decidedly critical analysis. Two able scholars have executed a splendid and useful text.

The definitions of the trigonometric functions based on the right triangle precede the general definitions. This procedure is much simpler for the beginner in trigonometry. The preliminary definitions, the usual initial simple formulas, and simple identities are given in Chapter I. Chapter II is devoted to the solution of the right triangle by the natural tables. In this chapter a brief—but illuminating—treatment is given on computing with approximate numbers. Since all measurements are approximate, each student of science should become sensitive in the use of approximate numbers as concerns the number of figures to be retained and the range or error. At the end of Chapter II a varied collection of forty-three problems is given from geometry, surveying, and mechanics.

Chapters III, IV, V, and VI cover the general definitions of the functions, the reduction formulas, identities, and radian measure. Chapter VII gives a splendid treatment of the graphical representation of the functions—more ample than is usually found in a trigonometry text. The addition formulas, their related formulas, identities, and logarithms constitute the material of chapters VIII and IX. Logarithms are used for the solution of the right triangle and the general triangle in Chapter X. In this chapter a wealth of problems is given, problems following each case and forty-three miscellaneous problems at the end of the chapter.

The text proper closes with chapters XI and XII in which a critical treatment is given of the inverse functions and trigonometric equations. Sixty-six pages of tables appear in the text.

A text in trigonometry which carries the tone of completeness and rigor attached to this work is justified on the market from time to time. This contention may be supported, in part, on the ground that mathematics departments must periodically change text books and a good one should be available which is up to date. The authors have pursued the practice of giving ample hints in connection with exercises.

The mechanical make-up of the text is excellent with clear and pleasing figures.

Louisiana Polytechnic Institute. P. K. SMITH.

The Laws of Order and Chaos. G. A. Linhart. Occasional Papers of Riverside Junior College. Riverside, California. 1937.

As the sub-title states, this paper is a "revival of some ancient concepts in general dynamics". It will be the purpose of this reviewer to describe some of the highlights of the paper, rather than to attempt to evaluate them critically.

Pressure is defined as "the agency which tends to bring about and maintain order" and interference as "the agency which tends to disrupt order and bring about chaos". Progress is defined as the expenditure of labor with respect to time. Then:

$$dW = PdS$$

$$dW/dP = Q$$

$$dS = Qd(\log P)$$

where W is work, P is pressure, S is interference and Q is progress. The statement is made that "these three relations are necessary and entirely sufficient, in a general way, both for the practical evaluation and for the theoretical interpretation of any dynamic process in the realm of science or in the field of commerce."

The assumption that "in any continuous process the instantaneous rate of change of the progress with respect to the interference is proportional to the progress still to be attained" leads to the equation:

$$\log[Q/(Q_0 - Q)] = K \log P + \log k$$

where Q_0 is the "amount of progress at the start, or at the completion, of the process, depending upon whether the progress is to be negative or positive."

The *Summary* calls attention to the fact that the universe came into existence before mathematical formulas, and "hence, the several equations developed here are only statistical representations of every day phenomena in a world of progress and interference".

The remainder of the paper applies the foregoing general considerations to the specific cases of the gas laws and heat capacity

Here some historical background is presented and several equations are deduced, the most important of which are:

$$V_0/(V-V_0)=kP^K$$

$$K-1=1/bT^a$$

where V_0 is the ultimate molal volume, a and b are constants characteristic of the given substance and k and K vary with the temperature.

In general, the paper is very thought-provoking, and is recommended to those who are interested in fundamental scientific hypotheses.

Santa Monica Junior College. L. J. ADAMS.

Calculus. By J. V. McKelvey. Macmillan, New York, 1937. ix+420 pages.

This text, which is made up of twenty chapters and a table of integrals (102), includes the usual discussions found in a first course of differential and integral calculus. One chapter is devoted to each of the following topics: singular points of algebraic curves, infinite series and differential equations.

As a whole the book is carefully written. The author has succeeded in presenting his material in an interesting manner without omitting any of the essential basic concepts. The chapter on definite integrals is especially worthy of note. However, the reviewer notes a few topics which might be improved. The rules given for maximum and minimum points and inflection points do not cover the possibility of derivatives changing sign by undergoing a discontinuity. The fundamental theorem of integration is presented under the assumption that the independent variable subdivisions are equal. Only one short article is devoted to hyperbolic functions. The rule given for integrating exact differential equations does not always hold (see Cohen, *Differential Equations*, page 11, Heath, 1906). In several problems and in some integration formulas necessary restrictions have been omitted. Problem 25, page 25, is obviously misprinted.

Virginia Military Institute. WILLIAM E. BYRNE.